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TRIGONOMETRY,
P L A N E
A N D
S P H E R I C A L;
WITH THE
CONSTRUCTION and APPLICATION
O F
L O G A R I T H M S.

By *THOMAS SIMPSON*, F.R.S.

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Plane Trigonometry.



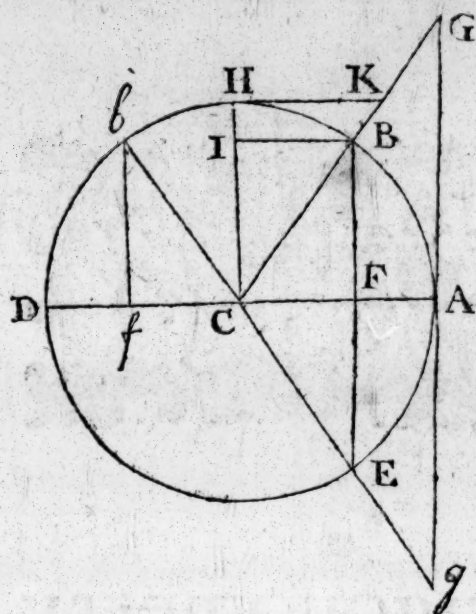
DEFINITIONS.

1. **P**LANE trigonometry is the art whereby, having given any three parts of a plane triangle, (except the three angles) the rest are determined. In order to which, it is not only requisite that the peripheries of circles, but also certain right-lines in, and about, the circle, be supposed divided into some assigned number of equal parts.

2. The periphery of every circle is supposed to be divided into 360 equal parts, called degrees; and each degree into 60 equal parts, called minutes; and each minute into 60 equal parts, called seconds, or second minutes, &c.

B 2

3. Any



3. Any part AB of the periphery of the circle is called an arch, and is said to be the measure of the angle ACB at the center, which it subtends.

Note, The degrees, minutes, seconds, &c. contained in any arch, or angle, are wrote in this manner, $50^{\circ} 18' 35''$, which signifies that the given arch, or angle, contains 50 degrees, 18 minutes, and 35 seconds.

4. The difference of any arch from 90° (or a quadrant) is called its complement; and its difference from 180° (or a semicircle) its supplement.

5. A chord, or subtense, is a right-line drawn from one extremity of an arch to the other: thus the right line BE is the chord, or subtense, of the arch BAE or BDE.

6. The sine, or right-sine, of an arch, is a right-line drawn from one extremity of the arch, perpendicular to the diameter passing through the other extremity. Thus BF is the sine of the arch AB or DB.

7. The versed sine of an arch is the part of the diameter intercepted between the sine and the periphery. Thus AF is the versed sine of AB; and DF of DB.

8. The

8. The co-sine of an arch is the part of the diameter intercepted between the center and sine; and is equal to the sine of the complement of that arch. Thus CF is the co sine of the arch AB, and is equal to BI, the sine of its complement HB.

9. The tangent of an arch is a right-line touching the circle in one extremity of that arch, produced from thence till it meets a right line passing through the center and the other extremity. Thus AG is the tangent of the arch AB.

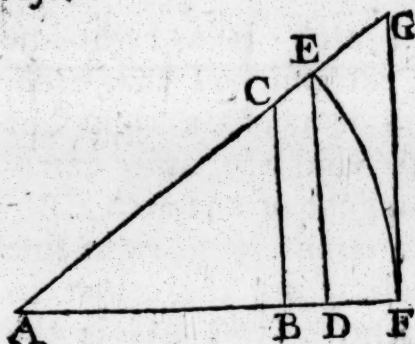
10. The secant of an arch is a right-line reaching, without the circle, from the center to the extremity of the tangent. Thus CG is the secant of AB.

11. The co-tangent, and co-secant, of an arch are the tangent, and secant, of the complement of that arch. Thus HK and CK are the co-tangent and co-secant of AB.

12. A trigonometrical canon is a table exhibiting the length of the sine, tangent, and secant, to every degree and minute of the quadrant, with respect to the radius; which is supposed unity, and conceived to be divided into 10000, or more, decimal parts. By the help of this table, and the doctrine of similar triangles, the whole business of trigonometry is performed; which I shall now proceed to shew. But, first of all, it will be proper to observe, that the sine of any arch Ab greater than 90° , is equal to the sine of another arch AB as much below 90° ; and that, its co sine Cf, tangent Ag, and secant Cg, are also respectively equal to the co-sine, tangent, and secant of its supplement AB; but only are negative, or fall on contrary sides of the points C and A, from whence they have their origin. All which is manifest from the definitions.

THEOREM I.

In any right-angled plane triangle ABC, it will be as the hypotenuse is to the perpendicular, so is the radius (of the table) to the sine of the angle at the base.



For, let AE or AF be the radius to which the table of sines, &c. is adapted, and ED the sine of the angle A or arch EF (*Vid. Def. 3. and 6.*); then, because of the similar triangles ACB and AED, it will be $AC : BC :: AE : ED$ (*by 14. 4.*) *Q. E. D.*

Thus, if $AC = ,75$, and $BC = ,45$; then it will be, $,75 : ,45 :: 1$ (radius) : the sine of $A = 6$; which, in the table, answers to $36^\circ 52'$, the measure, or value of A.

THEOREM II.

In any right-angled plane triangle ABC, it will be, as the base AB is to the perpendicular BC, so is the radius (of the table) to the tangent of the angle at the base.

For, let AE or AF be the radius of the table, or canon (*see the preceding figure*), and FG the tangent of the angle A, or arch EF (*Vid. Def. 3. and 9.*); then, by reason of the similarity of the triangles ABC, AFG, it will be, $AB : BC :: AF : FG$. *Q. E. D.*

Note, In the quotations where you meet with two numbers (as 14. 4.) without any mention of Prop. Theor. &c. reference is made to the second edition of the Elements of Geometry published by the same author; to which this little tract is designed as an Appendix.

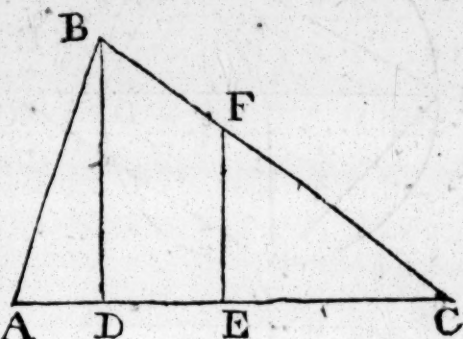
Thus

Thus let $AB = ,8$, and $BC = ,5$; then we shall have $,8 : ,5 :: 1$ (radius) : tangent $A = ,625$; whence A itself is found, by the canon, to be $32^{\circ} 00'$.

THEOREM III.

In every plane triangle ABC, it will be, as any one side is to the sine of its opposite angle, so is any other side to the sine of its opposite angle.

For take $CF = AB$, and upon AC let fall the perpendiculars BD and FE ; which will be the sines of the angles A and C to the equal radii AB and CF . Now the triangles

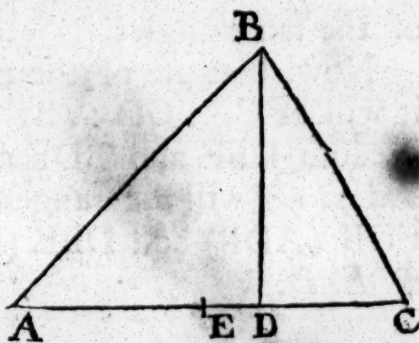


CBD , CFE being similar, we have $CB : BD$ ($\sin. A$) :: CF (AB) : FE ($\sin. C$). Q. E. D.

THEOREM IV,

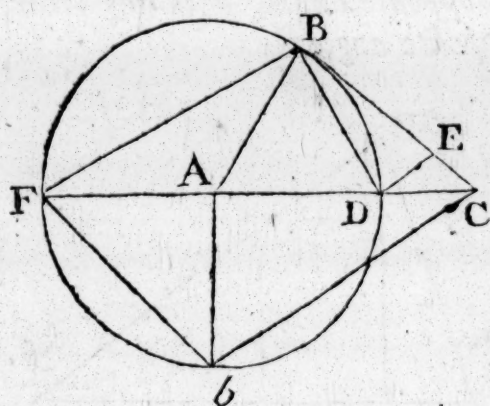
As the base of any plane triangle ABC, is to the sum of the two sides, so is the difference of the sides to twice the distance DE of the perpendicular from the middle of the base.

For (by Cor. to 9. 2.)
 $\overline{AB + BC} \times \overline{AB - BC}$
 $= AC \times 2DE$;
 whence $AC : AB + BC :: AB - BC : 2DE$ (by 10. 4.) Q. E. D.



THEOREM V.

In any plane triangle, it will be, as the sum of any two sides is to their difference, so is the tangent of half the sum of the two opposite angles, to the tangent of half their difference.



For, let ABC be the triangle, and AB and AC the two proposed sides; and from the center A, with the radius AB, let a circle be described, intersecting CA produced, in D and F; so that CF may express the sum, and CD the difference, of the sides AC and AB: join F, B and B, D, and draw DE parallel to FB, meeting BC in E.

Then, because $2\angle ADB = \angle ADB + \angle ABD$ (by 12. 1.) $= \angle C + \angle ABC$ (by 9. 1.) it is plain that $\angle ADB$ is equal to half the sum of the angles opposite to the sides proposed. Moreover, since $\angle ABC = \angle ABD (\angle ADB) + \angle DBC$, and $\angle C = \angle ADB - \angle DBC$ (by 9. 1.) it is plain that $\angle ABC - \angle C$ is $= 2\angle DBC$; or that $\angle DBC$ is equal to half the difference of the same angles.

Now, because of the parallel lines BF and ED, it will be $CF : CD :: BF : DE$; but BF and DE, because DBF and BDE are right-angles (by 13. 3. and 7. 1.) will be tangents of the foresaid angles FDB ($\angle ADB$) and DBE ($\angle DBC$) to the radius BD. Q. E. D.

COROL.

COROLLARY.

Hence, in two triangles ABC and AbC, having two sides equal, each to each, it will be (*by equality*), as tang. $\frac{AbC + ACb}{2}$: tang. $\frac{AbC - ACb}{2}$::

tang. $\frac{ABC + ACB}{2}$: tang. $\frac{ABC - ACB}{2}$. But,

if CA**b** be supposed a right-angle, then will AbC + ACb also = a right angle (*by Cor. 3. to 10. 1.*) and

the tangent of $\frac{AbC + ACb}{2}$ = radius. Therefore in this case our proportion will become,

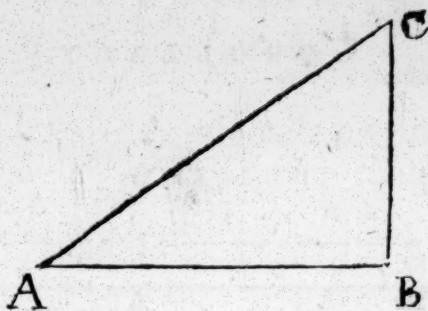
Radius : tang. $\frac{AbC - ACb}{2}$ (= AbC - 45°) ::

tang. $\frac{ABC + ACB}{2}$: tang. $\frac{ABC - ACB}{2}$. Which

gives the following *Theorem*, for finding the angles opposite to any two proposed sides; the included angle, and the sides themselves, being known.

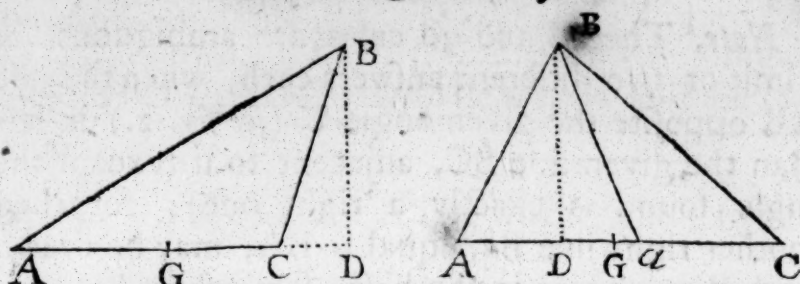
As the lesser of the proposed sides (Ab or AB) is to the greater (AC), so is radius to the tangent of an angle (AbC, see Theor. 2.) And as radius to the tangent of the excess of this angle above 45°, so is the tangent of half the sum of the required angles to the tangent of half their difference.*

* This Theorem, though it requires two proportions, is commonly used by Astronomers in determining the elongation and parallaxes of the planets (being best adapted to logarithms); for which reason it is here given.



The solutions of the cases of right-angled plane triangles.

Case	Given	Sought	Solution.
1	The hyp. AC and the angles	One leg BC	As radius is to the sine of A, so is the hyp. AC to the leg BC (<i>by Theor. I.</i>)
2	The hyp. AC and one leg BC	The angles	As AC : BC :: radius : sin. A (<i>Theor. I.</i>) whose complement is the angle C.
3	The hyp. AC and one leg BC	The other leg AB	Let the angles be found, <i>by Case 2.</i> and then the required leg AB, <i>by Case 1.</i>
4	The angles and one leg BC	The hyp. AC	As sine A : radius :: the leg BC : to the hyp. AC (<i>Theor. I.</i>)
5	The angles and one leg BC	The other leg AB	As sine A : BC :: sine C : AB (<i>by Theor. III</i>) Or, as radius : tang. C :: BC : AB (<i>by Theor. II.</i>)
6	The two legs AB and BC	The angles	As AB : BC :: radius : tang. A (<i>by Theorem II.</i>) whose complement is the angle C.
7	The two legs AB and BC	The hyp. AC	Let the angles be found, <i>by Case 6.</i> and then the hyp. AC, <i>by Case 4.</i>



The solution of the cases of oblique plane triangles.

Case	Given	Sought	Solution.
1	The angles and one side AB	Either or the other sides BC	As $\sin C : AB :: \sin A : BC$ (by Theor. III.)
2	Two sides AB, BC and an ang. C op. to one of 'em.	The other angles A and ABC	As $AB : \sin C :: BC : \sin A$ (by Theor. III.) which added to C, and the sum subtracted from 180 gives the other angle ABC.
3	Two sides AB, BC and an opp. angle C	The other side AC	Let the angle ABC be found, by the preceding case, and then it will be, $\sin C : AB :: \sin ABC : AC$ (by Theor. III.)
4	Two sides AC, AB and the included angle A	The other angles C and ABC	As sum of AB and AC : their diff. :: tang. of half the sum of ABC and C : tang. of half their diff. (by Theor. V.) which added to, and subtracted from, the half sum, gives the two angles.
5	Two sides AC, AB and the incl. $\angle A$.	The other side BC	Let the angles be found by the last case; and then BC, by Case 1.
6	All the three sides	An angle, suppose A	Let fall a perp. BD opp. to the req. angle: then (by Theor. IV.) as $AC : \text{sum of } AB \text{ and } BC :: \text{their diff.} : \text{dist. } DG \text{ of the perp. from the middle of the base; whence, } AD \text{ being also known, the angle } A \text{ will be found by Case 2. of right angles.}$

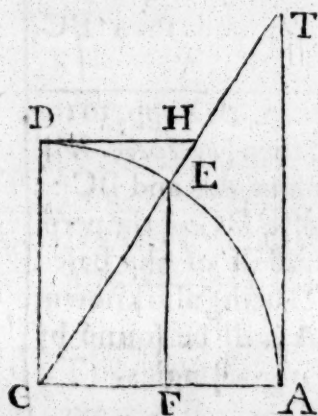
Note,

Note, The 2d and 3d cases are ambiguous, or admit of two different answers each, when the side AB opposite the given angle C (*see fig. 2.*) is less than the given side BC, adjacent to it (except the angle found is exactly a right one): for then another right-line Ba, equal to BA, may be drawn from B to a point in the base, somewhere between C and the perpendicular BD, and therefore the angle found by the proportion $AB (aB) : \sin. C :: BC : \sin. A$ (or of CaB ,) may, it is evident, be either the acute angle A, or the obtuse one CaB (which is its supplement), the sines of both being exactly the same.

Having laid down the method of resolving the different cases of plane triangles, by a table of sines and tangents; I shall here shew the manner of constructing such a table (as the foundation upon which the whole doctrine is grounded); in order to which, it will be requisite to premise the following propositions.

PROPOSITION I.

The sine of an arch being given, to find its co-sine, versed sine, tangent, co-tangent, secant, and co-secant.



Let AE be the proposed arch, EF its sine, CF its co-sine, AF its versed sine, AT its tangent, CT its secant, DH its co-tangent, and CH its co-secant.

Then (*by 8. 2.*) we have $CF = \text{the square root of } CE^2 - EF^2$; whence, not only the co-sine CF, but also the versed sine AF, will

will be known. Then because of the similar triangles CFE, CAT, and CDH, it will be (by 14. 4.)

1. $CF : FE :: CA : AT$; whence the tangent is known.

2. $CF : CE (CA) :: CA : CT$; whence the secant is known.

3. $EF : CF :: CD : DH$; whence the co-tangent is known.

4. $EF : EC (CD) :: CD : CH$; whence the co-secant is known.

Hence it appears,

1. That the tangent is a fourth proportional to the co-sine, the sine, and radius.

2. That the secant is a third-proportional to the co-sine and radius.

3. That the co-tangent is a fourth proportional to the sine, co-sine, and radius.

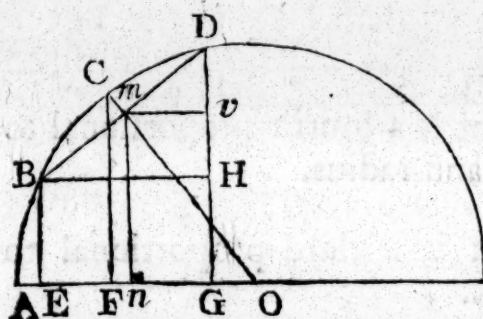
4. And that the co-secant is a third proportional to the sine and radius.

5. It appears moreover (because $AT : AC :: CD (AC) : DH$), that the rectangle of the tangent and co-tangent is equal to the square of the radius (by 10. 4.): whence it likewise follows, that the tangent of half a right-angle is equal to the radius; and that the co-tangents of any two different arches (represented by P and Q) are to one another, inversely as the tangents of the same arches: for, since $\text{tang. } P \times \text{co-tang. } P = \text{squ. rad.} = \text{tang. } Q \times \text{co-tang. } Q$; therefore will co-tang.

tang. P : co-tang. Q :: tang. Q : tang. P ; or as
co-tang. P : tang. Q :: co-tang. Q : tang. P (by
10. 4.)

PROP. II.

If there be three equidifferent arches AB, AC, AD, it will be, as radius is to the co-sine of their common difference BC, or CD, so is the sine CF, of the mean, to half the sum of the sines BE + DG, of the two extremes: and, as radius to the sine of the common difference, so is the co-sine FO of the mean, to half the difference of the sines of the two extremes.



For let BD be drawn, intersecting the radius OC in *m*; also draw *mn* parallel to CF, meeting AO in *n*; and BH and *mv*, parallel to AO,

meeting DG in H and *v*.

Then, the arches BC and CD being equal to each other (*by hypothesis*), OC is not only perpendicular to the chord BD, but also bisects it (*by 1. 3.*) and therefore B*m* (*or Dm*) will be the sine of BC (*or DC*), and Om its co-sine: moreover *mn*, being an arithmetical mean between the sines BE, DG of the two extremes (because B*m* = D*m*) is therefore equal to half their sum, and D*v* equal to half their difference. But, because of the similar triangles OCF, Om*n* and D*vm*,

It will be $\left\{ \begin{array}{l} OC : Om :: CF : mn \\ OC : Dm :: FO : Dv \end{array} \right\} \text{ Q. E. D.}$

COROL.

COROLLARY I.

Because of the foregoing proportions, we have

$$mn \left(\frac{DG+BE}{2} \right) = \frac{Om \times CF}{OC}^*, \text{ and } Dv \left(\frac{DG-BE}{2} \right)$$

$$= \frac{Dm \times FO}{OC}; \text{ and therefore } DG+BE = \frac{2Om \times CF}{OC}$$

 and $DG-BE = \frac{2Dm \times FO}{OC}.$

COROLLARY II.

Hence, if the mean arch AC be supposed that of 60° ; then OF being the co-sine of 60° , = sine $30^\circ = \frac{1}{2}$ chord of $60^\circ = \frac{1}{2} OC$, it is manifest that $DG-BE$ will, in this case, be barely = Dm ; and consequently $DG = Dm + BE$. From whence, and the preceding corollary, we have these two useful theorems.

1. If the sine of the mean, of three equidifferent arches (supposing radius unity) be multiplied by twice the co-sine of the common difference, and the sine of either extreme be subtracted from the product, the remainder will be the sine of the other extreme.

2. The sine of any arch, above 60 degrees, is equal to the sine of another arch, as much below 60° , together with the sine of its excess above 60° .

* Note, $\frac{Om \times CF}{OC}$, taken in a geometrical sense, denotes a fourth-proportional to OC , Om and CF ; but, arithmetically, it signifies the quantity arising by dividing the product of the measures of Om and CF by that of OC . Understand the like of others.

PROP.

PROP. III.

To find the sine of a very small arch; suppose that of $15'$.

It is found, in p. 181. of the Elements, that the length of the chord of $\frac{1}{384}$ of the semi-periphery is expressed by ,00818121 (radius being unity); therefore, as the chords of very small arches are to each other nearly as the arches themselves (*vid.* p. 181.) we shall have, as $\frac{1}{384} : \frac{1}{360} :: ,00818121 : ,008726624$, the chord of $\frac{1}{360}$, or half a degree; whose half, or ,004363312, is therefore the sine of $15'$, very nearly.

From whence the sine of any inferior arch may be found by bare proportion. Thus, if the sine of $1'$ be required, it will be, $15' : 1' :: ,004363312 : ,000290888$, the sine of the arch of one minute, nearly.

But if you would have the sine of $1'$ more exactly determined (from which the sines of other arches may be derived with the same degree of exactness); then let the operations, in p. 181, be continued to 11 bisections, and a greater number of decimals be taken; by which means you will get the chord of $\frac{1}{6144}$ part of the semi-periphery to what accuracy you please: then, by proceeding as above (for finding the sine of $15'$), the sine of 1 minute will also be obtained to a very great degree of exactness.

PROP. IV.

To shew the manner of constructing the trigonometrical canon.

First, find the sine of an arch of one minute, by the preceding Prop. and then its co-sine, by Prop:

Prop. 1. which let be denoted by C; then (by Theor. 1. p. 13.) we shall have

$$2C \times \text{fine } 1' - \text{fine } 0' = \text{fine } 2'.$$

$$2C \times \text{fine } 2' - \text{fine } 1' = \text{fine } 3'.$$

$$2C \times \text{fine } 3' - \text{fine } 2' = \text{fine } 4'.$$

$$2C \times \text{fine } 4' - \text{fine } 3' = \text{fine } 5'.$$

And thus are the fines of 6', 7', 8', &c. successively derived from each other.

The fines of every degree and minute, up to 60°, being thus found; those of above 60° will be had by addition only (from Theor. 2. p. 15.) then, the fines being all known, the tangents and secants will likewise become known, by Prop. 1.

Note, If the sine of every 5th minute, only, be computed according to the foregoing method, the fines of all the intermediate arches may be had from thence, by barely taking the proportional parts of the differences, and that so near as to give the first six places true in each number; which is sufficiently exact for all common purposes.

SCHOLIUM.

Although what has been hitherto laid down for constructing the trigonometrical canon, is abundantly sufficient for that purpose, and is also very easily demonstrated; yet, as the first sine, from whence the rest are all derived, must be carried on to a great number of places, to render the numerous deductions from it but tolerably exact (because in every operation the error is multiplied), I shall here subjoin a different method, which will be found to have the advantage, not only in that, but in many other respects.

First, then, from the co-sine of 15°, which is given (by p. 181 of the Elements) $= \frac{1}{2} \sqrt{2 + \sqrt{3}}$
 $= .965925826$, &c. ($= \frac{1}{2}$ the supplement chord of 30°) and the sine of 18°, which is $= \frac{1}{2} \sqrt{\frac{5}{4}} - \frac{1}{4} =$
C
,309017,

,309017, &c. (equal to half the side of a decagon inscribed in the circle) let the co-sine of 3° , the difference between 18° and 15° , be found *; from which the co-sine of $45'$ will be had, by two bisections only: whence the sines of all the arches in the progression $1^\circ 30'$, $2^\circ 15'$, $3^\circ 00'$, $3^\circ 45'$, &c. may be determined (by *Theor.* 1. p. 15.) and that to any assigned degree of exactness.

The sines of all the terms of the progression $45'$, $1^\circ 30'$, $2^\circ 15'$, &c. up to 60° , being thus derived, the next thing is to find, by help of these, the sines of all the intermediate arches, to every single minute.

This, if you desire no more than the 4 or 5 first places of each (which is exact enough where nothing less than degrees and minutes is regarded), may be effected by barely taking the proportional parts of the differences.

But if a greater degree of accuracy be insisted on, and you would have a table carried on to 7 or 8 places, each number (which is sufficient to give the value of an angle to seconds, and even to thirds, in most cases, then the operation may be as follows:

1°. Multiply the sum of the sines of any two adjacent terms of the progression $45'$, $1^\circ 30'$, $2^\circ 15'$, $3^\circ 00'$, $3^\circ 45'$, &c. (betwixt which you would find all the intermediate sines) by the fraction ,0000000423, for a first product; and this, again, by 22, for a second product; to which last, let $\frac{1}{45}$ of the difference of the two proposed sines (or extremes) be added, and the sum will be the excess of the first of the intermediate sines above the lesser extreme.

* Note, *The co-sine of the difference of two arches* (supposing radius unity), is found by adding the product of their sines to that of their co-sines; as is hereafter demonstrated.

2°. From

2°. From this excess let the first product be continually subtracted; that is, first, from the excess itself; then from the remainder; then from the last remainder, and so on 44 times.

3°. To the lesser extreme add the forementioned excess; and, to the sum, add the first remainder; to this sum add the next remainder, and so on continually: then the several sums thus arising will respectively exhibit the sines of all the intermediate arches, to every single minute, exclusive of the last; which, if the work be right, will agree with the greater extreme itself, and therefore will be of use in proving the operation.

But to illustrate the matter more clearly, let it be proposed to find the sines of all the intermediate arches between $3^{\circ} 00'$ and $3^{\circ} 45'$ to every single minute, those of the extremes being given, from the foregoing method, equal to ,05233595 and ,06540312 respectively. Here, the sum of the sines of the extremes being multiplied by ,0000000423, the first product will be ,00000000498, &c. or ,0000000050, *nearly* (which is sufficiently exact for the present purpose); and this, again, multiplied by 22, gives ,00000011 for a 2d product; which added to ,0002903815, $\frac{1}{43}$ part of the difference of the two given extremes, will be ,0002904915, the excess of the sine of $3^{\circ} 01'$ above that of $3^{\circ} 00'$. From whence, by proceeding according to the 2d and 3d rules, the sines of all the other intermediate arches are had, by addition and subtraction only. See the operation.

Construction of the Table

,000290	4915 excess	,05233595	fine 3° 0'
,000000	0050	,0002904915	
4865	1 st rem.	,0526264415	fine 3° 1'
50		2904865	
4815	2 ^d rem.	,0529169280	fine 3° 2'
50		2904815	
4765	3 ^d rem.	,0532074095	fine 3° 3'
50		2904765	
4715	4 th rem.	,0534978860	fine 3° 4'
50		2904715	
4665	5 th rem.	,0537883575	fine 3° 5'
50		2904665	
4615	6 th rem.	,0540788240	fine 3° 6'
50		2904615	
4565	7 th rem.	,0543692855	fine 3° 7'
50		2904565	
4515	8 th rem.	,0546597420	fine 3° 8'
50		2904515	
4465	9 th rem.	,0549501935	fine 3° 9'
50		2904465	
4415	10 th rem.	,0552406400	fine 3° 10'
&c.		&c.	

Again, as a second example, let it be required to find the sines of all the arches, to every minute, between 59° 15' and 60° 00'; those of the two extremes being first found, by the preceding method.

of Sines, Tangents and Secants. 21

method. In this case, the two extremes, being ,85940641 and ,86602540, their sum will be = 1,72543, &c. and their difference = ,00661899; whereof the former, multiplied by ,0000000423 (*see the rule*) gives ,00000007298, &c. or ,0000000730, nearly, for the first product (which is exact enough for our purpose); therefore the 2d product, or ,0000000730 $\times$ 22, will be ,0000016060; which, added to $\frac{1}{45}$ of the difference, gives ,0001486947; from whence the operation will be as follows:

,0001486947 excess	,85940641	line 59° 15'
,0000000730 1 st prod.	0001486947	
86217 1 st rem.	,859551047	line 59° 16'
730	1486217	
85487 2 ^d rem.	,8597037264	line 59° 17'
730	1485487	
84757 3 ^d rem.	,8598522751	line 59° 18'
730	1484757	
84027 4 th rem.	,8600007508	line 59° 19'
730	1484027	
83297 5 th rem.	,8601491535	line 59° 20'
730	1483297	
82567 6 th rem.	,8602974832	line 59° 21'
	1482567	
&c.	,8604457399	line 59° 22'
	&c.	

After the same manner the sines of all the intermediate arches between any other two proposed extremes may be derived, even up to 90 degrees;

but those of above 60° are best found from those below, as has been shewn elsewhere.

The reasons upon which the foregoing operations are founded, depend upon principles too foreign from the main design of this treatise, to be explained here, (even would room permit); however, as to the correctness and utility of the method itself, I will venture to affirm, that, whoever has the inclination, either to calculate new tables, or to examine those already extant, will not find one quarter of the trouble, this way, as he unavoidably must according to the common methods.



Spherical Trigonometry.

DEFINITIONS.

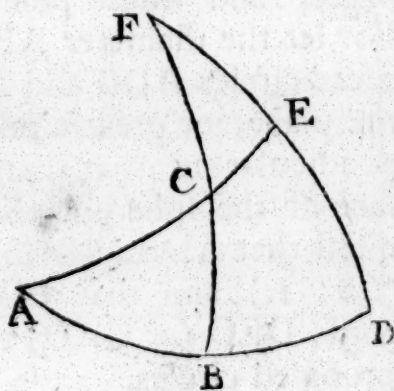
1. **A** Great-circle of a sphere is a section of the sphere by a plane passing thro' the center.

2. The axis of a great-circle is a right line passing through the center, perpendicular to the plane of the circle: and the two points, where the axis intersects the surface of the sphere, are called the poles of the circle.

3. A spherical angle is the inclination of two great-circles.

4. A spherical triangle is a part of the surface of the sphere included by the arches of three great-circles; which arches are called the sides of the triangle.

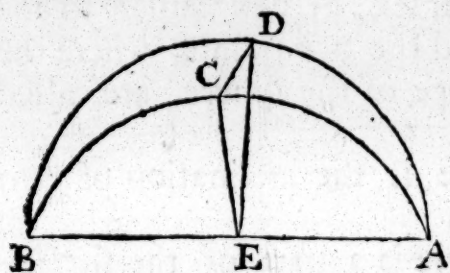
5. If thro' the poles A and F of two great-circles DF and DA, standing at right-angles, two other great-circles ACE and FCB be conceived to pass, and thereby form two spherical triangles ABC and FCE, the latter of the triangles so formed is said to be the complement of the former; and *vice versa*.



COROLLARIES.

1. It is manifest (*from Def. 1.*) that the section of two great-circles (as it passes through the center) will be a diameter of the sphere; and consequently, that their peripheries will always intersect each other in two points at the distance of a semi-circle, or 180 degrees.

2. It also appears (*from Def. 2.*) that all great-circles, passing through the pole of a given circle, cut that circle at right-angles; because they pass through, or coincide with the axis, which is perpendicular to it.

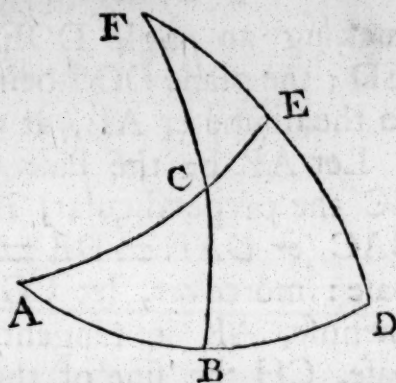


3. It follows moreover, that the periphery of a great-circle is every where 90 degrees distant from its pole; and that the measure of a spherical angle CAD* is an arch of a great-circle intercepted by the two circles ACB, ADB forming that angle, and whose pole is the angular point A. For let the diameter AB be the intersection of the great-circles ADB and ACB (*see Corol. 1.*) and let the plane, or great-circle, DEC be conceived perpendicular to that diameter, intersecting the surface of the sphere in the arch CD; then it is manifest that $AD = BD = 90^\circ$, and $AC = BC = 90^\circ$ (*Cor. 1.*) and that CD is the measure of the angle DEC (*or* CAD) the inclination of the two proposed circles.

* Note, Although a spherical angle is, properly, the inclination of two great-circles, yet it is commonly expressed by the inclination of their peripheries at the point where they intersect each other.

4. Hence

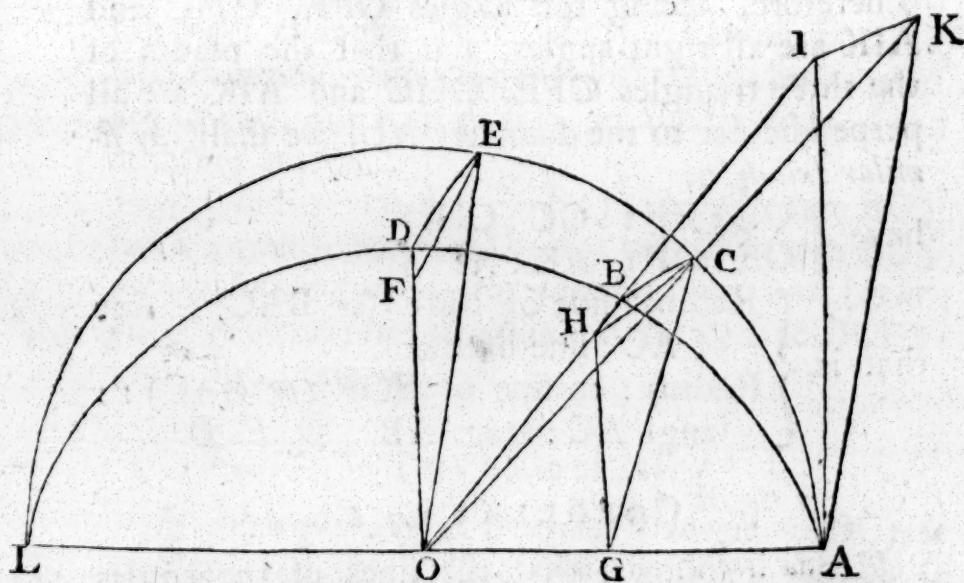
4. Hence it is also manifest, that the angles B and E, of the complementary triangles ABC and FCE, are both right angles; and that CE is the complement of AC, CF of BC, BD (or the angle F) of AB and EF of ED (or the angle A).



THEOREM I.

In any right-angled spherical triangle it will be, as radius is to the sine of the angle at the base, so is the sine of the hypotenuse to the sine of the perpendicular; and as radius to the co-sine of the angle at the base, so is the tangent of the hypotenuse to the tangent of the base.

DEMONSTRATION.



Let ADL and AEL be two great-circles of the sphere intersecting each other in the diameter AL, making

making an angle DOE, measured by the arch ED; the plane DOE being supposed perpendicular to the diameter AL, at the center O.

Let AB be the base of the proposed triangle, BC the perpendicular, AC the hypotenuse, and BAC (or DAE = DE = DOE) the angle at the base: moreover, let CG be the sine of the hypotenuse, AK its tangent, AI the tangent of the base, CH the sine of the perpendicular, and EF the sine of the angle at the base; and let I, K and G, H be joined.

Because CH is perpendicular to the plane of the base (or paper), it is evident, that the plane GHC will be perpendicular to the plane of the base, and likewise perpendicular to the diameter AL, because GC, being the sine of AC, is perpendicular to AL. Moreover, since both the planes OIK and AIK are perpendicular to the plane of the base (or paper), their intersection IK will also be perpendicular to it, and consequently the angle AIK a right-angle. Therefore, seeing the angles OFE, GHC and AIK are all right-angles, and that the planes of the three triangles OFE, GHC and AIK are all perpendicular to the diameter AL, we shall, by similar triangles,

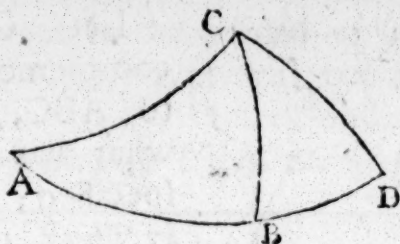
have $\left\{ \begin{array}{l} OE : EF :: GC : CH \\ OE : OF :: AK : AI \end{array} \right\}$

that is, $\left\{ \begin{array}{l} \text{Radius : sine of EOF (or BAC) :: sine of AC : sine of BC.} \\ \text{Radius : co-sine of EOF (or BAC) ::} \\ \text{tang. AC : tang. AB. } \textit{Q. E. D.} \end{array} \right.$

COROLLARY. I.

Hence it follows, that the sines of the angles of any oblique spherical triangles ADC are to one another, directly, as the sines of the opposite sides.

For



For let BC be perpendicular to AD;
 then $\left\{ \begin{array}{l} \text{radius : sine A :: sine AC : sine BC} \\ \text{sine : radius :: sine D :: sine DC : sine BC} \end{array} \right\}$ *by the*
former part of the theorem; we shall have, $\text{sine A} \times$
 $\text{sine AC} (= \text{radius} \times \text{sine BC}) = \text{sine D} \times \text{sine DC}$
(by 10. 4.) and consequently $\text{sine A} : \text{sine D} :: \text{sine}$
 $\text{DC} : \text{sine AC}$; or $\text{sine A} : \text{sine DC} :: \text{sine D} : \text{sine}$
 AC .

COROLLARY 2.

It follows, moreover, that, in right-angled
 spherical triangles ABC, DBC, having one leg
 BC common, the tangents of the hypotenuses
 are to each other, inversely, as the co-sines of the
 adjacent angles.

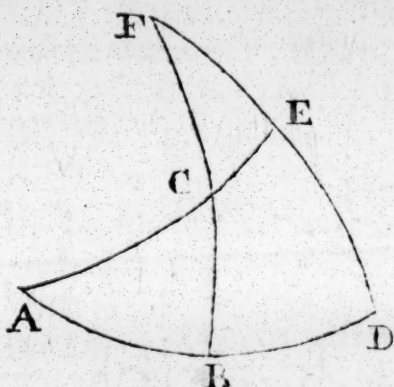
For $\left\{ \begin{array}{l} \text{radius : co-sine ACB :: tan. AC : tan. BC} \\ \text{sine : radius :: co-sine DCB :: tan. DC : tan. BC} \end{array} \right\}$
by the latter part of the theorem; we shall (*by ar-*
guing as above) have $\text{co-sine ACB} : \text{co-sine DCB} ::$
 $\text{tang. DC} : \text{tang. AC}$.

THEOREM II.

In any right-angled spherical triangle (ABC) it
will be, as radius is to the co-sine of one leg, so is
the co-sine of the other leg to the co-sine of the hy-
pothenuse.

DEMONSTRATION

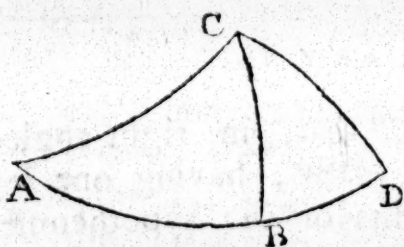
DEMONSTRATION.



Let CEF be the complementary triangle to ABC, according to what has been already specified; then it will be, by *Theor. 1. Case 1.*

Radius : sine F :: sine CF : sine CE ; that is,
 Radius : co-sine BA :: co sine CB : co-sine AC
 (*see Cor. 4. p. 25.*) Q. E. D.

COROLLARY.



Hence, if two right-angled spherical triangles ABC, CBD have the same perpendicular BC, the co-sines of their hypotenuses will be to each other, directly, as

the co sines of their bases.

For $\left\{ \begin{array}{l} \text{rad} : \text{co-sin. BC} :: \text{co-sin. AB} : \text{co-sine AC,} \\ \text{since } \left\{ \begin{array}{l} \text{rad} : \text{co-sin. BC} :: \text{co-sin. DB} : \text{co-sine DC,} \end{array} \right. \end{array} \right.$
 therefore, by equality and permutation, co sine
 AB : co-sine DB :: co sine AC : co-sine DC.

THEOREM III.

In any right-angled spherical triangle (ABC) it will be, as radius is to the sine of either angle, so is the co-sine of the adjacent leg to the co-sine of the opposite angle.

DEMON-

DEMONSTRATION.

Let CEF be as in the preceding proposition; then, by *Theor. 1. Case 1.* it will be, radius : sine C :: sine CF : sine EF; that is, radius : sine C :: co-sine BC : co-sine A. *Q. E. D.*

COROLLARY.

Hence, in right-angled spherical triangles ABC, CBD, having the same perpendicular BC (*see the last figure*), the co-sines of the angles at the base will be to each other, directly, as the sines of the vertical angles :

For { radius : sine BCA :: co-sine CB : co-sine A,
since { radius : sine BCD :: co-sine CB : co-sine D,
therefore, by equality and permutation,

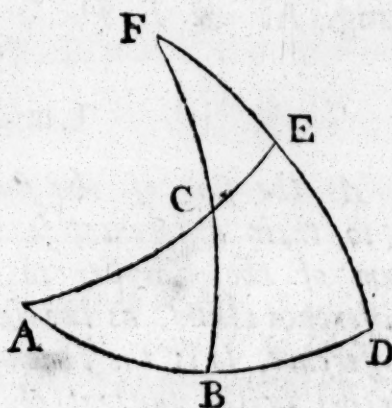
Co-sine A : co-sine D :: sine BCA : sine BCD.

THEOREM IV.

In any right-angled spherical triangle (ABC) it will be, as radius is to the sine of the base, so is the tangent of the angle at the base to the tangent of the perpendicular.

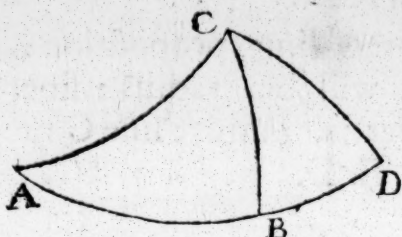
For, supposing CEF
as before,

it will be, as radius :
co sine of F :: tang.
CF : tang. FE (*by the
latter part of Theor. 1.*)
that is, radius : sine
AB :: co-tang. BC :
co-tang. A :: tang. A :
tang. BC (*by Corol. 5.
p. 13.*) *Q. E. D.*



COROL-

COROLLARY.



Hence it follows, that, in right angled spherical triangles ABC, DBC, having the same perpendicular BC, the sines of the bases will be to each other, *inversely*, as the tangents of the angles at the bases :

For $\left\{ \begin{array}{l} \text{radius} : \text{fine AB} :: \text{tang. A} : \text{tang. BC} \\ \text{radius} : \text{fine DB} :: \text{tang. D} : \text{tang. BC} \end{array} \right\}$
 since } we shall (*by reasoning as in Cor. 1. Theor. 1.*) have
 Sine AB : sine DB :: tang. D : tang. A.

THEOREM V.

In any right-angled spherical triangle it will be, as radius is to the co-sine of the hypotenuse, so is the tangent of either angle to the co-tangent of the other angle.

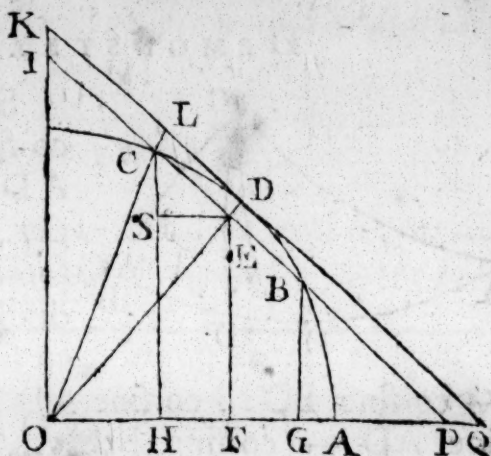
For (CEF being as in the last) it will be, as radius : sine CE :: tang. C : tang. EF (by Theorem 4.) that is, radius : co-sine AC :: tang. C : co-tang. A. Q. E. D.

LEMMA.

As the sum of the sines of two unequal arches is to their difference, so is the tangent of half the sum of those arches to the tangent of half their difference : and, as the sum of the co-sines is to their difference, so is the co-tangent of half the sum of the arches to the tangent of half the difference of the same arches.

For,

For, let AB and AC be the two proposed arches, and let BG and CH be their sines, and OG and OH their co-sines : moreover, let the arch BC be equally divided in D, so that CD may be half the difference; and AD



half the sum, of AB and AC : let the radii OD and OC be drawn, and also the chord CB, meeting OE in E and OA (produced) in P ; draw ES parallel to AO, meeting CH in S, and EF and OK perpendicular to AO, and let the latter meet EC (produced) in I ; lastly, draw QDK perpendicular to OD, meeting OA, OC and OI (produced) in Q, L and K.

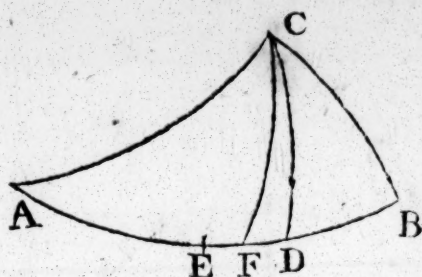
Because $CD = BD$, it is manifest that OD is not only perpendicular to the chord BC, but bisects it in E ; whence, also, EF bisects HG ; and therefore $CH + BG = 2EF$, and $CH - BG = 2CS$; also $OG + OH = 2OF$, and $OG - OH = 2HF$: but $2EF(CH + BG) : 2CS(CH - BG) :: EF : CS :: EP : EC$ (by 14. 4.) $:: DQ$ (the tangent of AD) : DL (the tangent of DC, by 20. 4.) And $2OF(OG + OH) : 2HF(OG - OH) :: EI : EC :: DK$ (the co-tang. of AD) : DL (the tang. DC). Q. E. D.

THEOREM VI.

In any spherical triangle ABC it will be, as the co-tangent of half the sum of the two sides is to the tangent of half their difference, so is the co-tangent of half the base to the tangent of the distance (DE) of the perpendicular from the middle of the base.

DEMON-

DEMONSTRATION.



Since co-sine AC : co-sine BC :: co-sine AD : co-sine BD (by *Cor. to Theor. 2.*) therefore, by composition and division, co-sine AC + co-sine BC : co-sine AC — co-sine BC :: co-sine AD + co-sine BD : co-sine AD — co-sine BD. But (by the preceding lemma) co-sine AC + co-sine BC : co-sine AC — co-sine BC :: co-tang. $\frac{AC+BC}{2}$: tang. $\frac{AC-BC}{2}$; and co-sine AD + co-sine BD : co-sine AD — co-sine BD :: co-tang. of AE $\left(\frac{AD+BD}{2}\right)$: tang. DE $\left(\frac{AD-BD}{2}\right)$; whence, by equality, co-tang. $\frac{AC+BC}{2}$: tang. $\frac{AC-BC}{2}$:: co-tang. AE : tang. DE.

COROLLARY.

Since the last proportion by permutation, becomes co-tang. $\frac{AC+BC}{2}$: co-tang. AE :: tang. $\frac{AC-BC}{2}$: tang. DE, and it is proved, in *p. 13.* that the tangents of any two arches are, inversely, as their co-tangents; it follows, therefore, that tang. AE : tang. $\frac{AC+BC}{2}$:: tang. $\frac{AC-BC}{2}$: tang. DE; or, that the tangent of half the base, is to the tangent of half the sum of the sides, as the tangent of half the difference of the sides, to the tangent

tangent of the distance of the perpendicular from the middle of the base.

THEOREM VII.

In any spherical triangle ABC, it will be, as the co-tangent of half the sum of the angles at the base, is to the tangent of half their difference, so is the tangent of half the vertical angle, to the tangent of the angle which the perpendicular CD makes with the line CF bisecting the vertical angle. (See the preceding figure.)

DEMONSTRATION.

It will be (by Corol. to Theor. 3.) co-sine A : co-sine B :: sine ACD : sine BCD; and therefore, co-sine A + co-sine B : co-sine A — co-sine B :: sine ACD + sine BCD : sine ACD — sine BCD. But (by the lemma) co-tang. $\frac{B+A}{2}$: tang. $\frac{B-A}{2}$:: (co-sine A + co-sine B : co-sine A — co-sine B :: sine ACD + sine BCD : sine ACD — sine BCD) tang. ACF : tang. DCF. Q. E. D.



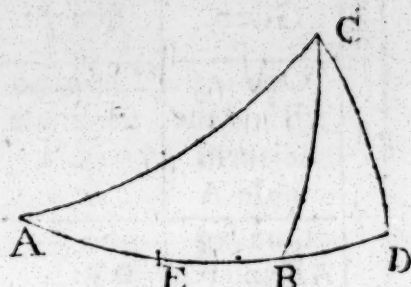
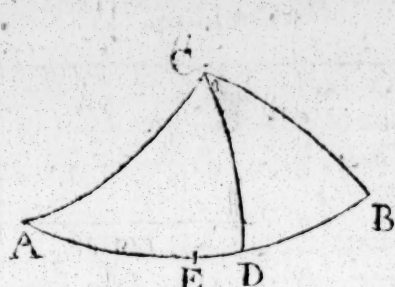
The solution of the cases of right-angled spherical triangles.

Case	Given	Sought	Solution.
1	The hyp. AC and one angle A	The opposite leg BC	As radius : fine hyp. AC :: fine A : fine BC (<i>by the former part of Theor. 1.</i>)
2	The hyp. AC and one angle A	The adjacent leg AB	As radius : co-fine of A :: tang. AC : tang. AB (<i>by the latter part of Theor. 1.</i>)
3	The hyp. AC and one angle A	The other angle C	As radius : co-fine of AC :: tang. A : co-tang. C (<i>by Theor. 5.</i>)
4	The hyp. AC and one leg AB	The other leg BC	As co-fine AB : radius :: co-fine AC : co-fine BC (<i>by Theor. 2.</i>)
5	The hyp. AC and one leg AB	The opposite angle C	As fine AC : radius :: fine AB : fine C (<i>by the former part of Theor. 1.</i>)
6	The hyp. AC and one leg AB	The adjacent angle A	As tang. AC : tang. AB :: radius : co-fine A (<i>by Theor. 1.</i>)
7	One leg AB and the adjacent angle A	The other leg BC	As radius : fine AB :: tangent A : tangent BC (<i>by Theor. 4.</i>)

Case

Case	Given.	Sought.	Solution.
8	One leg AB and the adjacent angle A	The opposite angle C	As radius : fine A :: co-fine of AB : co-fine of C (<i>by Theor. 3.</i>)
9	One leg AB and the adjacent angle A	The hyp. AC	As co-fine of A : radius :: tang. AB : tang. AC (<i>by Theor. 1.</i>)
10	One leg BC and the opposite angle A	The other leg AB	As tang. A : tang. BC :: radius : fine AB (<i>by Theor. 4.</i>)
11	One leg BC and the opposite angle A	The adjacent angle C	As co-fine BC : radius :: co-fine of A : fine C (<i>by Theor. 3.</i>)
12	One leg BC and the opposite angle A	The hyp. AC	As fin. A : fin. BC :: radius : fine AC (<i>by Theor. 1.</i>)
13	Both legs AB and BC	The hyp. AC	As radius : co-fine AB :: co-fine BC : co-fine AC (<i>by Theor. 2.</i>)
14	Both legs AB and BC	An angle, suppose A	As fine AB : radius :: tang. BC : tang. A (<i>by Theor. 4.</i>)
15	Both angles A and C	A leg, suppose AB	As fin. A : co-fine C :: radius : co-fine AB (<i>by Theor. 3.</i>)
16	Both angles A and C	The hyp. AC	As tang. A : co-tang. C :: radius : co-fine AC (<i>by Theor. 5.</i>)

Note, The 10th, 11th and 12th cases are ambiguous; since it cannot be determined, by the data, whether AB, C, and AC, be greater or less than 90 degrees each.



The solution of the cases of oblique spherical triangles:

Case	Given	Sought	Solution.
1	Two sides AC, BC and an angle A opposite to one of them	The angle B opposite to the other	As fine BC : fine A :: fine AC : fine B (<i>by Cor. 1. to Theor. 1.</i>) <i>Note</i> , This case is ambiguous when BC is less than AC; since it cannot be determined from the data whether B be acute or obtuse.
2	Two sides AC, BC and an angle A opposite to one of them	The included angle ACB	Upon AB produced (if need be) let fall the perpendicular CD; then (<i>by Theor. 5.</i>) rad. : co-fine AC :: tang. A : co-tang. ACD; but (<i>by Cor. 2. to Theor. 2.</i>) as tang. BC : tang. AC :: co-fine ACD : co-fine BCD. Whence $ACB = ACD + BCD$ is known.
3	Two sides AC, BC and an angle opposite to one of them	The other side AB	As rad. : co-fine A :: tang. AC : tang. AD (<i>by Theor. 1.</i>) and (<i>by Cor. to Theor. 2.</i>) as co-fine AC : co-fine BC :: co-fine AD : co-fine BD. <i>Note</i> , This and the last case are both ambiguous when the first is so.
4	Two sides AC, AB and the included angle A	The other side BC	As rad. : co-fine A :: tang. AC : tan. AD (<i>by Theor. 1.</i>) whence BD is also known; then (<i>by Corol. to Theor. 2.</i>) as co-fine AD : co-fine BD :: co-fine AC : co-fine BC.
5	Two sides AC, AB and the included angle A	Either of the other angles, suppose B	As rad. : co-fine A :: tang. AC : tang. AD (<i>by Theor. 1.</i>) whence BD is known; then (<i>by Cor. to Theor. 4.</i>) as fine BD : fine AD :: tang. A : tang. B.

Case

Case	Given	Sought	Solution.
6	Two angles A, ACB and the side AC betwixt them.	The other angle B	As rad. : co-fine AC :: tang. A : co-tang. ACD (by Theor. 5.) whence BCD is also known; then (by Cor. to Theor. 3.) as fine ACD : fine BCD :: co-fine A : co-fine B.
7	Two angles A, ACB and the side AC betwixt them	Either of the other sides, suppose BC	As rad. : co-fine AC :: tang. A : co-tang. ACD (by Theor. 5.) whence BCD is also known; then as co-fine of BCD : co-fine ACD :: tan. AC : tang. BC (by Cor. 2. to Theor. 1.)
8	Two angles A, B and a side AC opposite to one of them.	The side BC opposite the other	As fine B : fine AC :: fine A : fine BC (by Cor. 1. to Theor. 1.)
9	Two angles A, B and a side AC opposite to one of them	The side AB betwixt them	As rad. : co-fine A :: tang. AC : tang. AD (by Theor. 1.) and as tan. B : tang. A :: fine AD : fine BD (by Cor. to Theor. 4.) whence AB is also known.
10	Two angles A, B and a side AC opposite to one of them	The other angle ACB	As rad. : co-fine AC :: tang. A : co-tang. ACD (by Theor. 5.) and as co-fine A : co-fine B :: fine ACD : fine BCD (by Cor. to Theor. 3.) whence ACB is also known.
11	All the three sides AB, AC and BC,	An angle, suppose A	As tang. $\frac{1}{2}AB$: tang. $\frac{AC+BC}{2}$:: tang. $\frac{AC-BC}{2}$: tang. DE, the distance of the perpendicular from the middle of the base (by Cor. to Theor. 6.) whence AD is known; then, as tang. AC : tang. AD :: rad. : co-fine A (by Theor. 1.)
12	All the three angles A, B and ACB	A side, suppose AC	As co-tan. $\frac{ABC+A}{2}$: tan. $\frac{ABC-A}{2}$:: tang. $\frac{ACB}{2}$: tang. of the angle included by the perpendicular and a line bisecting the vertical angle; whence ACD is also known; then (by Theor. 5.) tang. A : co-tang. ACD :: rad. : co-fine AC.

Note, In letting fall your perpendicular, let it always be from the end of a given side and opposite to a given angle.

Of the nature and construction of logarithms, with their application to the doctrine of triangles.

AS the business of trigonometry is wonderfully facilitated by the application of logarithms; which are a sett of artificial numbers, so proportioned among themselves and adapted to the natural numbers 2, 3, 4, 5, &c. as to perform the same things by addition and subtraction, *only*, as these do by multiplication and division: I shall here, for the sake of the young beginner (for whom this small tract is chiefly intended) add a few pages upon this subject. But, first of all, it will be necessary to premise something, in general, with regard to the indices of a geometrical progression, whereof logarithms are a particular species.

Let, therefore, 1, a , a^2 , a^3 , a^4 , a^5 , a^6 , a^7 , &c. be a geometrical progression whose first term is unity, and common ratio any given quantity a . Then it is manifest,

1. *That, the sum of the indices of any two terms of the progression is equal to the index of the product of those terms.* Thus $2 + 3$ ($= 5$) is $=$ the index of $a^2 \times a^3$, or a^5 ; and $3 + 4$ ($= 7$) is $=$ the index of $a^3 \times a^4$, or a^7 . This is universally demonstrated in p. 19. of my book of *Algebra*.

2. *That, the difference of the indices of any two terms of the progression is equal to the index of the quotient of one of them divided by the other.* Thus

$5 - 3$ is $=$ the index of $\frac{a^5}{a^3}$ or a^2 . Which is only the converse of the preceding article.

3. *That*

3. That, the product of the index of any term by a given number (n) is equal to the index of the power whose exponent is the said number (n). Thus 2×3 (6) is = the index of a^2 raised to the 3d power (or a^6). This is proved in p. 38, and also follows from article 1.

4. That, the quotient of the index of any term of the progression by a given number (n) is equal to the index of the root of that term defined by the same number (n). Thus $\frac{6}{3}$ (2) is = the index of (a^2) the cube root of a^6 . Which is only the converse of the last article.

These are the properties of the indices of a geometrical progression; which being universally true, let the common ratio be now supposed indefinitely near to that of equality, or the excess of a above unity, indefinitely little; so that some term, or other, of the progression $1, a, a^2, a^3, a^4, a^5, \&c.$ may be equal to, or coincide with, each term of the series of natural numbers $2, 3, 4, 5, 6, 7, \&c.$ Then are the indices of those terms called logarithms of the numbers to which the terms themselves are equal. Thus, if $a^m = 2$, and $a^n = 3$, then will m and n be logarithms of the numbers 2 and 3 respectively.

Hence it is evident, that what has been above specified, in relation to the properties of the indices of powers, is equally true in the logarithms of numbers; since logarithms are nothing more than the indices of such powers as agree in value with those numbers. Thus, for instance, if the logarithms of 2 and 3 be denoted by m and n ; that is, if $a^m = 2$, and $a^n = 3$, then will the logarithm of 6, (the product of 2 and 3) be equal to $m + n$ (agreeable to article 1); because 2×3 (6) = $a^m \times a^n = a^{m+n}$.

But we must now observe, that there are various forms or species of logarithms; because it is evident that what has been hitherto said, in respect to the properties of indices, holds equally true in relation to any equimultiples, or like parts, of them; which have, manifestly, the same properties and proportions, with regard to each other, as the indices themselves. But the most simple kind of all, is *Neiper's*, otherwise called the *hyperbolic*.

The hyperbolic logarithm of any number is the index, of that term of the logarithmic progression agreeing with the proposed number, multiplied by the excess of the common ratio above unity.

Thus, if e be an indefinite small quantity, the hyperbolic logarithm of the natural number agreeing with any term $\overline{1 + e^n}$ of the logarithmic progression $1, 1 + e, \overline{1 + e^2}, \overline{1 + e^3}, \overline{1 + e^4}, \&c.$ will be expressed by ne .

PROPOSITION I.

The hyperbolic logarithm (L) of a number being given, to find the number itself, answering thereto.

Let $\overline{1 + e^n}$ be that term of the logarithmic progression $1, \overline{1 + e^1}, \overline{1 + e^2}, \overline{1 + e^3}, \overline{1 + e^4}, \&c.$ which is equal to the required number (N). Then, because $\overline{1 + e^n}$ is, universally, $= 1 + ne + n \cdot \frac{n-1}{2} \cdot e^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot e^3 \&c.$ we shall, also, have $1 + ne + n \cdot \frac{n-1}{2} \cdot e^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot e^3 \&c. = N$. But, because n (from the nature of logarithms) is here supposed indefinitely great, it is evident, first, that the numbers connected to it by the sign $-$, may be rejected, as far as any assigned number

number of terms being indefinitely small in comparison of n : it is also evident, that they may be rejected in all the rest of the terms of the series; because these terms (by reason of the indefinite smallness of e) bear no assignable proportion to the preceding ones. Hence we have $1 +$

$$ne + \frac{n^2 e^2}{2} + \frac{n^3 e^3}{2.3} + \frac{n^4 e^4}{2.3.4} \&c. = N: \text{ but } ne \text{ is}$$

$$(\equiv L) \text{ the hyperbolic logarithm of } \overline{1 + e|^n} \text{ (or } N) \text{ by what has been already specified: therefore } 1 +$$

$$L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} \&c. = N.$$

Q. E. I.

PROP. II.

To determine the hyperbolic logarithm (L) of any given number (N).

It appears from the preceding Prop. that $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} \&c.$ is $= N$: therefore, if $x + 1$ be put $= N$, we shall have $L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} \&c. = x$; and consequently, by reverting the series, $L = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} \&c.$

Q. E. I.

OTHERWISE.

Because $\overline{1 + e|^n} = N$ (by the definition of logarithms) we shall have $1 + e = N^{\frac{1}{n}} = \overline{1 + x|^m}$; by putting $1 + x = N$, and $m = \frac{1}{n}$. Therefore,

$$\overline{1 + x|^m} \text{ being } = 1 + mx + m \cdot \frac{m-1}{2} \cdot x^2 + m \cdot$$

$$m-1$$

$\frac{m-1}{2} \cdot \frac{m-2}{3} \cdot x^3$ &c. we have $e = mx + m$.

$\frac{m-1}{2} \cdot x^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot x^3$ &c. where, m

being rejected in the factors $m-1$, $m-2$, $m-3$ &c. as indefinitely small in comparison of 1, 2, 3

&c. the equation will become $e = mx - \frac{mx^2}{2} +$

$\frac{mx^3}{3} - \frac{mx^4}{4}$ &c. whence $\frac{e}{m} (= ne = L) = x -$

$\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ &c. the very same as before.

But this series, tho' indeed the most easy and natural, is of little use in determining the logarithms of large numbers; since, in all such cases, it diverges, instead of converging. It will be proper, therefore, to give, here, the invention of other methods, which authors have had recourse to, in order to obtain a series that will always converge.

First, then, let the number whose logarithm you would find be denoted by $\frac{1}{1-x}$; where it is manifest (however great that number may be) x will be always less than unity: moreover, let $1+e^n$ (as before) be the term of the logarithmic progression agreeing with the proposed number, or, which is the same, let $1+e^n = \frac{1}{1-x}$: then, by taking the root on both sides) we

shall have $1+e = \frac{1}{1-x} = \frac{1}{1-x}^{\frac{1}{n}} = \frac{1}{1-x}^{\frac{m}{n}}$

(by

(by making $m = -\frac{1}{n}$) $= 1 - mx + m \cdot \frac{m-1}{2}$.

$x^2 - m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot x^3$ &c. where m being re-

jected in the factors $m-1$, $m-2$, &c. (as before) our equation will become $1 + e = 1 - mx -$

$\frac{mx^2}{2} - \frac{mx^3}{3}$ &c. whence $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$ &c.

$= \frac{e}{-m} = ne =$ the hyperbolic logarithm of $\frac{1}{1-x}$;

Which series, it is manifest, will always converge,

let the value of $\frac{1}{1-x}$ be ever so great; because x will be always less than unity.

But it is further observable that this series has exactly the same form (except in its signs) with that above for the logarithm of $1+x$; and that, if both of them be added together, the series $2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7}$ &c. thence arising, will be more

simple than either of them; since one half of the terms will be intirely destroyed thereby. Therefore, because the sum of the logarithms of any two numbers is equal to the logarithm of the product of those numbers, (*see Article 1.*) it is mani-

fest that $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$ &c. will truly express

the logarithm of $\frac{1+x}{1-x}$, or $\frac{1-x}{1} \times \frac{1}{1+x}$. Which

series converges, still, faster than $x + \frac{x^2}{2} + \frac{x^3}{3}$ &c.

not only because the even powers are here destroyed, but because x , in finding the logarithm of any given number (N), will have a less value.

But

But now, to determine what this value must be, make $\frac{1+x}{1-x} = N$, and then x will be found $= \frac{N-1}{N+1}$; but if the quantity proposed be a fraction $\left(\frac{P}{Q}\right)$, instead of a whole number, make $\frac{P}{Q} = \frac{1+x}{1-x}$,

and you will have $x = \frac{P-Q}{P+Q}$: either of which values, substituted in the foregoing series $2x + \frac{2x^3}{3} + \frac{2x^5}{5} \&c.$ will give the hyperbolic logarithm of the respective number.

Example. Let it be proposed to find the hyperbolic logarithm of the number 2.

Here x being $= \frac{2-1}{2+1} = \frac{1}{3}$, and $x^2 = \frac{1}{9}$; we shall have

$$\begin{array}{ll}
 x \left(= \frac{1}{3} \right) & = ,33333333 \&c. \\
 x^3 \left(= \frac{1}{27} \right) & = ,037037037 \&c. \\
 x^5 \left(= \frac{1}{243} \right) & = ,004115226 \&c. \\
 x^7 \left(= \frac{1}{2187} \right) & = ,000457247 \&c. \\
 x^9 \left(= \frac{1}{19683} \right) & = ,000050805 \&c. \\
 x^{11} \left(= \frac{1}{177147} \right) & = ,000005645 \&c. \\
 x^{13} \left(= \frac{1}{1594323} \right) & = ,000000627 \&c. \\
 x^{15} \left(= \frac{1}{14348907} \right) & = ,000000069 \&c. \\
 & \&c. \qquad \qquad \&c.
 \end{array}$$

Which values being respectively divided by the numbers, 1, 3, 5, 7, 9, &c. and the several quotients added together, (*see the general series*) we shall have ,346573590 &c. whose double, being ,693147180 &c. is the hyperbolic logarithm of the number 2.

After

After the very same manner the hyperbolic logarithm of any other number may be determined; but, as the series converges, slower and slower, the higher we go, it is usual, in computing of tables, to derive the logarithms we would find, by help of others already known; for which there are various methods; but the following is the most commodious and simple, that has occurred to me, especially, when a great degree of accuracy is required.

It is thus. Let a , b and c denote any three numbers in arithmetical progression, whose common difference is unity; then, a being $= b - 1$ and $c = b + 1$, we shall have $ac = b^2 - 1$, and consequently $\frac{b^2}{ac} = \frac{ac + 1}{ac}$. Whence, by the nature of logarithms, we likewise have $2 \log. b - \log. a - \log. c = \log. \frac{ac + 1}{ac}$: but the logarithm of $\frac{ac + 1}{ac}$, by putting $\frac{1}{2ac + 1} = x$, will be $\frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} \&c.$ (by what has been already shewn): which being denoted by S , we shall

$$\text{have } \left\{ \begin{array}{l} \log. b = \frac{1}{2} \log. a + \frac{1}{2} \log. c + \frac{1}{2} S. \\ \log. a = 2 \log. b - \log. c - S. \\ \log. c = 2 \log. b - \log. a - S. \end{array} \right\}$$

As an example hereof, let it be proposed to find the hyperbolic logarithm of 3.

Then, the hyperbolic logarithm of 2 being already found $= .693147180 \&c.$ that of 4, which is the double thereof, will also be known. Therefore, taking $a = 2$, $b = 3$, and $c = 4$, we shall, in this

this case, have $x \left(\frac{1}{2ac + 1} \right) = \frac{1}{17}$, and $x^3 = \frac{1}{289}$;
whence

$$x (= \frac{1}{17}) = ,058823529 \text{ \&c.}$$

$$x^3 (= \frac{x^3}{289}) = ,000203542 \text{ \&c.}$$

$$x^5 (= \frac{x^3}{289}) = ,000000704 \text{ \&c.}$$

&c.

&c.

Therefore $\frac{1}{2} S (x + \frac{x^3}{3} + \frac{x^5}{5} \text{ \&c.}) = ,058891517 \text{ \&c.}$ and

consequently hyp. log. 3. $\left(\frac{\text{hyp. log. } 2 + \text{hyp. log. } 4}{2} \right.$

$\left. + \frac{1}{2} S \right) = 1,098612288 \text{ \&c.}$

2. Let the hyperbolic logarithm of 10 be required.

The logarithms of 8 and 9 being given, from those of 2 and 3 (already found), *a* may, here, be

$= 8$, $b = 9$ and $c = 10$; and then $x \left(\frac{1}{2ac + 1} \right)$ being

$$= \frac{1}{161}, \text{ we shall have } \frac{1}{2} S \left(x + \frac{x^3}{3} + \frac{x^5}{5} \text{ \&c.} \right) = ,006211180 \text{ \&c.} + ,000000079 \text{ \&c. \&c.} = ,006211259 \text{ \&c.}$$

And therefore log. 10 $(2 \log. 9 - \log. 8 - S) = 2,302585092 \text{ \&c.}$

Hitherto we have had regard to logarithms of the hyperbolic kind: but those of any other kind may be derived from these, by, *barely*, multiplying by the proper multiplicator, or modulus.

Thus, in the *Brigean* (or common) form, where an unit is assumed for the logarithm of 10, the logarithm of any number will be found, by multiplying

tipling the hyperbolic logarithm of the same number by the fraction ,434294481 &c. which is the proper modulus of this form.

For, since the logarithms of all forms preserve the same proportion with respect to each other, it will be, as 2,302585092 &c. the hyperbolic log. of 10 (above found) is to (H) the hyperbolic logarithm of any other number, so is 1, the common logarithm of 10, to $\left(\frac{H}{2,302585092 \text{ \&c.}}\right)$ $H \times ,434294481 \text{ \&c.}$ the common logarithm of the same number.

But (to avoid a tedious multiplication, which will always be required when a great degree of accuracy is insisted on) the best way to find the logarithms of this form is from the series $2x + \frac{2x^3}{3} + \frac{2x^5}{5} \text{ \&c.} \times ,434294481 \text{ \&c.}$ which expresses

the common logarithm of $\frac{1+x}{1-x}$ (by what has been already shewn), and which, by making $R = ,868588963 \text{ \&c.}$ will stand more commodiously thus, $Rx + \frac{Rx^3}{3} + \frac{Rx^5}{5} + \frac{Rx^7}{7} \text{ \&c.}$

For an example hereof, let the common logarithm of 7 be required: in which case (the logarithms of 8 and 9 being known, from those of 2 and 3), we shall have $\log. 7 = 2 \log. 8 - \log. 9 - S$ (by the Theor.), S being $= Rx + \frac{Rx^3}{3} + \frac{Rx^5}{5} \text{ \&c.}$ ($=$ the common log. of $\frac{64}{63}$) and $x (= \frac{64-63}{64+63}) = \frac{1}{127}$: whence (x^2 being $= \frac{1}{16129}$) we shall have

Rx

$$Rx (= \frac{,8685 \text{ \&c.}}{127}) = ,006839283 \text{ \&c.}$$

$$Rx^3 (= \frac{Rx}{16129}) = ,000000424 \text{ \&c.}$$

$$Rx^5 (= \frac{Rx^3}{16129}) = ,00000000002 \text{ \&c.}$$

Consequently $S (Rx + \frac{Rx^3}{3} + \frac{Rx^5}{5} \text{ \&c.}) = ,006839424 \text{ \&c.}$ and $2 \log. 8 - \log. 9 - S = ,845098040 \text{ \&c.} =$ the common logarithm of 7 required. But the same conclusion may be brought out by fewer terms of the series, if the logarithms of the three first primes 2, 3 and 5 be supposed known; because those of 48 and 50 (which are composed of them) will likewise be known; from whence the logarithm of 7 ($= \frac{1}{2} \log. 49 = \frac{\log. 48 + \log. 50 + S}{4}$) will come out $= ,845098040 \text{ \&c.}$ (*as before*) which value will be true to 11 places of figures by taking the first term of the series, *only*.

Again, let the common logarithm of the next prime number, which is 11, be required. Here a may be taken $= 10$, $b = 11$, and $c = 12$; but fewer terms of the series will suffice, if other three numbers, composed of 11 and the inferior primes, be taken, whereof the common difference is an unit. Thus, because $98 = 2 \times 7 \times 7$, $99 = 3 \times 3 \times 11$ (9×11), and $100 = 2 \times 2 \times 5 \times 5$ (or 10×10), let there be taken $a = 98$, $b = 99$, and $c = 100$; and then, by the first term of the series only, the log. of 99 will be found true to 14 places; whence that of 11 ($\log. 99 - \log. 9$) is also known.

But

But notwithstanding all these artifices and compendiums, a method (similar to that in page 18.) for finding the logarithms of large numbers, one from another, by addition and subtraction, only, still seems wanting in the calculation of tables; I shall, therefore, here subjoin such a method.

1. Let A, B and C denote any three numbers in arithmetical progression, not less than 10000 each, whereof the common difference is 100.

2. From twice the logarithm of B, subtract the sum of the logarithms of A and C, and let the remainder be divided by 10000.

3. Multiply the quotient by 49,5, and to the product add $\frac{1}{1000}$ part of the difference of the logarithms of A and B; then the sum will be the excess of the logarithm of $A + 1$ above that of A.

4. From this excess let the quotient (found by Rule 2.) be continually subtracted, that is, first from the excess itself; then from the remainder, then from the next remainder, &c. &c.

5. To the logarithm of A add the said excess, and to the sum add the first of the remainders; to the last sum add the next remainder, &c. &c. then the several sums, thus arising, will exhibit the logarithms of $A + 1$, $A + 2$, $A + 3$, &c. respectively.

Thus, let it be proposed to find the logarithms of all the whole numbers between 17900 and 18100; those of the two extremes 17900 and 18100, and that of the mean (18000) being given.

E

Then

Then the logarithm of $\left\{ \begin{array}{l} A \\ B \\ C \end{array} \right\}$ being $\left\{ \begin{array}{l} 4,252853031 \\ 4,255272505 \\ 4,257678575 \end{array} \right\}$ equal to

we shall have $\frac{2 \log. B - \log. A - \log. C}{10000} =$

,00000000134 (see Rule 2.) which multiplied by 49,5, and the product added to $\frac{\log. B - \log. A}{100}$

gives ,00002426107 for the excess of the logarithm of $A + 1$ above that of A (by Rule 3.) From whence the work, being continued according to Rule 4 and 5, will stand as follows:

,000024	26107 excess.	4,25	2853031	log. 17900
	134		2426107	excess.
	25973 1 st rem.		287729207	log. 17901
	134		2425973	
	25839 2 ^d rem.		290155180	log. 17902
	134		2425839	
	25705 3 ^d rem.		292581019	log. 17903
	134		2425705	
	25571 4 th rem.		295006724	log. 17904
	134		2425571	
	25437 5 th rem.		297432295	log. 17905
	134		2425437	
	25303 6 th rem.		299857732	log. 17906
	134		2425303	
	25169 7 th rem.		302283035	log. 17907
	134		2425169	
	25035 8 th rem.		304708204	log. 17908
	134		2425035	
	24901 9 th rem.		307133239	log. 17909
	134		2424901	
	24767 10 th rem.		309558140	log. 17910
	&c. &c.		&c. &c.	

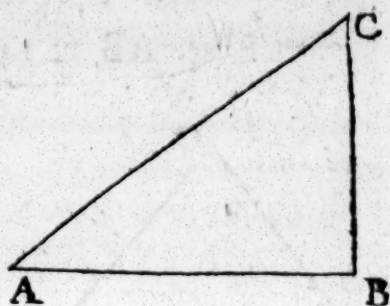
Note,

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Note, The logarithms found according to this method, in numbers between 10000 and 20000, are true to 8 or 9 places of figures: those of numbers between 20000 and 50000 err only in the 9th or 10th place; and those of above 50000 are true to 10 places, at least.

Having explained the manner of constructing a table of logarithms, and that by various methods, I now come to shew the use of such a table in the business of trigonometry.

First, in the right-angled plane triangle ABC, let there be given the hypotenuse AC = 17910 feet, and the angle A = 35° 20'; to find the perpendicular BC and the base AB.



Here, because radius : line 35° 20' :: 17910 : BC
(by *Theor.* 2. p. 6.) we have $BC = \frac{\text{line } 35^\circ 20' \times 17910}{\text{radius}}$;

therefore, because the addition and subtraction of logarithms answers to the multiplication and division of the natural numbers (*see* p. 38, 39.) we have $\log. BC = \log. \text{line } 35^\circ 20' + \log. 17910 - \log. \text{radius}$.

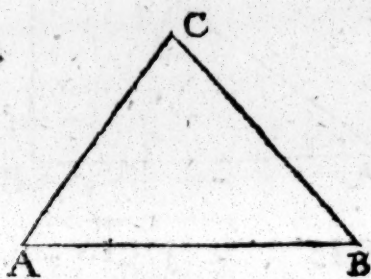
But, by the tables of artificial, or logarithmic, sines*, the log. line of 35° 20' will appear to be 9,7621775; to which add 4,2530956, the log. of 17910, and from the sum (14,0152731) take 10, the log. of radius, and there results 4,0152731 = the log. of BC; which, in the tables, answers to 10358, the length of BC required.

* A table of artificial sines is nothing more than a table of the logarithms of the numbers expressing the natural sines, to the radius 10000000000; whose logarithm is 10.

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Again, for AB, it will be, as radius : sine of C ($54^{\circ} 40'$) :: AC (17910) : AB (by *Theorem* 2.) Whence, by adding the logarithms of the second and third terms together and subtracting that of the first (as above), we have $AB = 14611$. See the operation.

$$\begin{array}{r} \text{Log. radius} - - - - 10, \\ \text{Log. sine C } (54^{\circ} 40') \quad 9,9115844 \\ \text{Log. AC } (17910) - - 4,2530956 \\ \hline \text{Log. AB} = 14611 - 4,1646800 \end{array}$$



Moreover, in the oblique plane triangle ABC, let there be given $AB = 75$, $AC = 60$, and the included angle $A = 48^{\circ}$; to find the other two angles. Then (by *Theorem* 5.) it will be,

$$\begin{array}{ll} \text{As } AB + AC \text{ (135)} & \text{its log. } 2,1303338 \\ \text{is to } AB - AC \text{ (15)} & \text{its log. } 1,1760913 \\ \text{so is T. } \frac{C+B}{2} \text{ (66^{\circ})} & - - - - 10,3514169 \end{array}$$

$$\hline 11,5275082$$

$$\text{to the T. } \frac{C-B}{2} = 14^{\circ} 00' \quad 9,3971744$$

Which 14° added to 66° , the half sum of the angles C and B, gives the greater $C = 80^{\circ}$; and subtracted therefrom, leaves the lesser $B = 52^{\circ}$.

Lastly,

Lastly, in the right-angled spherical triangle ABC, let there be given the hypotenuse $AC = 60^\circ$, and the angle $A = 23^\circ 29'$; to find the base and perpendicular. Then



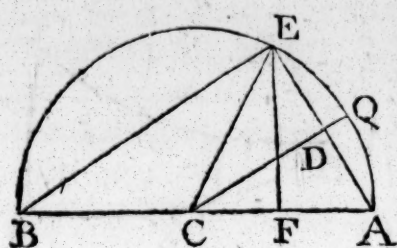
(by Theor. 1. p. 25.) the operation will be as follows :

Log. radius - - - - -	10,
Log. sine A ($23^\circ 29'$) - - - - -	9,6004090
Log. sine AC (60°) - - - - -	9,9375306
Log. sine BC = $20^\circ 11'$ - - - - -	9,5379396
Also, log. radius - - - - -	10,
Log. Co-f. A ($23^\circ 29'$) - - - - -	9,9624527
Log. T. AC (60°) - - - - -	10,2385606
Log. T. AB = $57^\circ 49'$ - - - - -	10,2010133

Having exhibited the manner of resolving all the common cases of plane and spherical triangles, both by logarithms and otherwise; I shall here subjoin a few propositions for the solution of the more difficult cases which sometimes occur; when, instead of the sides and angles themselves, their sums, or differences, &c. are given.

PROPOSITION I.

The sine, co-sine, or versed sine of an arch being given, to find the sine and co-sine, &c. of half that arch.



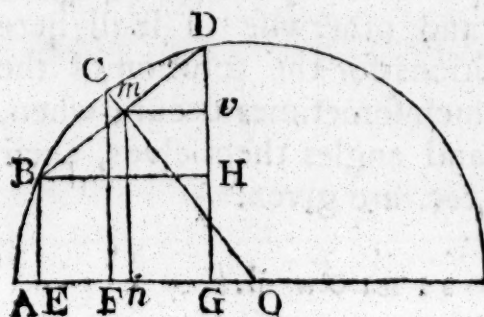
From the two extremes of the diameter AB, let the chords AE and BE be drawn, and let the radius CQ bisect AE, perpendicularly, in D (*Vid.* 1. 3.); then

will AD be the sine, and CD the co-sine, of the angle ACD, or $\frac{1}{2}$ ACE.

But $4AD^2 = AE^2$ (*by Cor. 1. to 6. 2.*) $= AB \times AF$ (*by Cor. to 19. 4.*) $= 2AC \times AF$; whence $AD^2 = \frac{1}{2} AC \times AF$: also $4CD^2 = BE^2 = AB \times BF = 2AC \times BF$; whence $CD^2 = \frac{1}{2} AC \times BF$. From which it appears, that the square of the sine of half any arch, or angle, is equal to a rectangle under half the radius and the versed sine of the whole; and that the square of its co-sine is equal to a rectangle under half the radius and the versed sine of the supplement of the whole arch, or angle.

PROP. II.

The sines and co-sines of two arches being given, to find the sines, and the co-sines, of the sum and difference of these arches.



Let AC and CD ($=BC$) be the two proposed arches; let CF and OF be the sine and co-sine of the greater AC, and let mD (Bm) and Om , be those of the lesser CD (or BC): moreover, let DG and OG be the sine and co-sine of the sum AD; and BE and OE, those of the difference AB. Draw mn parallel to CF, meeting AO in n ; also draw mv and

and BH parallel to AO, meeting GD in v and H: then it is plain, because $Dm = Bm$, that Dv is $= Hv$, and $mv = nG = En$; and that the triangles OCF, Omn and mDv are similar; whence we have the following proportions,

$$\left. \begin{array}{l} OC : Om :: CF : mn \\ OC : OF :: Dm : Dv \\ OC : OF :: Om : On \\ OC : CF :: Dm : mv \end{array} \right\} \text{whence} \left\{ \begin{array}{l} mn \times OC = Om \times CF. \\ Dv \times OC = Dm \times OF. \\ On \times OC = Om \times OF. \\ mv \times OC = Dm \times CF. \end{array} \right.$$

Now, by adding the two first of these equations together, we have $mn + Dv \times OC$ ($DG \times OC$) $= Om \times CF + Dm \times OF$; whence DG is known. Moreover, by taking the latter from the former, we get $mn - Dv \times OC$ ($BE \times OC$) $= Om \times CF - Dm \times OF$; whence BE is known.

In like manner, by adding the third and fourth equations together, we have $On + mv \times OC$ ($OE \times OC$) $= Om \times OF + Dm \times CF$; and, by subtracting the latter from the former, we have $On - mv \times OC$ ($OG \times OC$) $= Om \times OF - Dm \times CF$; whence OE and OG are also known. Q. E. I.

COROLLARY I.

Hence, if the sines of two arches be denoted by S and s ; their co-sines by C and c ; and radius by R ; then will

$$\text{the sine of their sum} = \frac{Sc + sC}{R},$$

$$\text{the sine of their difference} = \frac{Sc - sC}{R},$$

$$\text{the co-sine of their sum} = \frac{Cc - Ss}{R},$$

$$\text{the co-sine of their difference} = \frac{Cc + Ss}{R}.$$

COROLLARY II.

Hence, the sine of the double of either arch (when they are equal) will be $= \frac{2CS}{R}$, and its co-sine $= \frac{C^2 - S^2}{R}$; whence it appears, that *the sine of the double of any arch, is equal to twice the rectangle of the sine and co-sine of the single arch, divided by radius; and that its co-sine is equal to the difference of the squares of the sine and co-sine of the single arch, also, divided by radius.*

COROLLARY III.

Moreover, because $Dm \times CF = OC \times mv$ ($\frac{1}{2}OC \times EG = \frac{1}{2}OC \times OE - OG$); and $Om \times OF = OC \times On$ ($\frac{1}{2}OC \times 2On = \frac{1}{2}OC \times OE + OG$), it follows, *that the rectangle of the sines of any two arches (AC, CD (BC) is equal to a rectangle under half the radius, and the difference of the co-sines, of the sum and difference of those arches; and that the rectangle of their co-sines is equal to a rectangle under half the radius, and the sum of the co-sines, of the sum and difference of the same arches.*

PROP. III.

The tangents of two arches being given, to find the tangents of the sum, and difference, of those arches.

Let

Let AN and AM be the two arches, and AB and AC their tangents; also let NE be the tangent of their sum, in the first case, and the tangent of their difference, in the second, and let CF, perpendicular to the radius DN, be drawn: then, because of the equiangular triangles BAD and BFC, we shall have

$$\left\{ \begin{array}{l} BD \times CF = DA \times BC \\ BD \times BF = BA \times BC \end{array} \right\} \text{by 24. 3.}$$

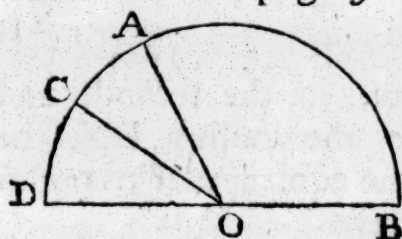
Take each of the last equal quantities from BD^2 , and there will remain $BD^2 - BD \times BF$ ($BD \times DF$) $= BD^2 - BA \times BC$: now $BD \times DF$ ($BD^2 - BA \times BC$): $BD \times CF$ ($DA \times BC$): DF : CF : DN (DA): NE : DA^2 : $DA \times NE$; whence, alternately, $BD^2 - BA \times BC$: DA^2 ($DA \times BC$: $DA \times NE$): BC : NE . But the first term, of this proportion, because $BD^2 = DA^2 + BA^2$, will also be expressed by $DA^2 + BA^2 - BA \times BC$, or by $DA^2 + BA^2 - BA \times AB \pm AC$; or, lastly, by, $DA^2 \mp BA \times AC$: therefore, the three first terms of the proportion being known, the fourth NE will likewise be known. Q. E. I.

COROLLARY.

Hence, if radius be supposed unity, and the tangents of two arches be denoted by T and t, it follows, that the tangent of their sum will be $= \frac{T + t}{1 - tT}$, and the tangent of their difference $= \frac{T - t}{1 + tT}$.

But,

But, it will be proper to take notice here (once for all) that, if in these, or any other theorems, the tangent, secant, co-sine, co-tangent, &c. of an arch greater than 90 degrees be concerned; then, instead thereof, the tangent, secant, co-sine, &c. of an arch, as much below 90 degrees, is to be taken, with a negative sign; according to the observation in page 5.



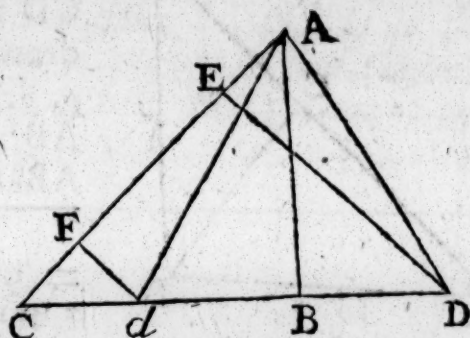
Thus, for instance, let BA be an arch greater than 90°, and let the tangent of the sum of AB and AC be required; supposing T to represent the tangent of AD (the supplement of AB) and t the tangent of AC: then, by writing $-T$ instead of T, in the first of the foregoing theorems, we shall have tang. of BC = $\frac{-T + t}{1 - t \times -T} = \frac{-T + t}{1 + tT}$, and therefore tang. DC ($-\text{tang. BC}$) = $\frac{T - t}{1 + tT}$; which is the very theorem demonstrated in the 2d case.

PROP. IV.

As the sum of the tangents of any two angles BAC, BAD, is to their difference, so is the sine of the sum of those angles, to the sine of their difference.

Let

Let BC and BD be the two proposed tangents, to the radius AB; take $Bd = BD$, join Ad , and draw DE and dF perpendicular to AC. It is manifest, be-



cause $Bd = BD$, that $Ad = AD$, and $dAB = DAB$, and, consequently, that CAd is the difference of the two angles BAC and BAD.

Now, by reason of the similar triangles CDE and CdF , it will be, $CD (CB + BD) : Cd (CB - BD) :: DE : dF$; but DE and dF are sines of DAE and dAF to the equal radii AD and Ad : whence the truth of the proposition is manifest.

COROLLARY.

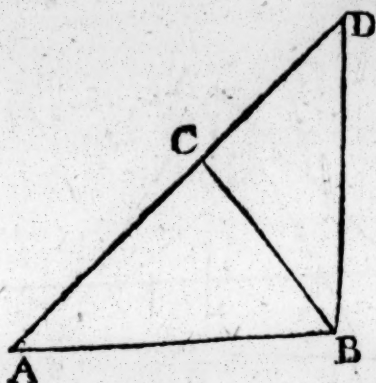
Hence it also appears, that the base (CD) of a plane triangle, is to (Cd) the difference of its two segments (made by letting fall a perpendicular), as the sine of the angle (CAD) at the vertex, to the sine of the difference of the angles at the base.

PROP. V.

In any plane triangle ABC, it will be, as the sum of the two sides plus the base, is to the sum of the two sides minus the base, so is the co-tangent of half either angle at the base, to the tangent of half the other angle at the base.

In

Properties of



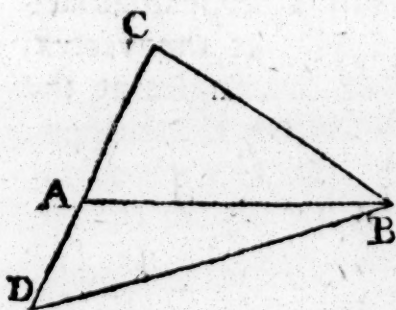
In AC produced, take
 $CD = BC$, and let BD be
 drawn: then (*by Theor. 5.*
p. 8.) it will be, $AD +$
 $AB : AD - AB :: \text{tang.}$

$$\frac{ABD + D}{2} \left(\frac{180^\circ - A}{2} \right. \\
= 90^\circ - \frac{1}{2} A) : \text{tang.} \\
\left. \frac{ABD - D}{2} \left(\frac{ABD - CBD}{2} \right. \right.$$

$\left. = \frac{ABC}{2} \right)$, that is, $AC + BC + AB : AC + BC$
 $- AB :: \text{co-tang. } \frac{1}{2} A : \text{tang. } \frac{1}{2} ABC.$ Q. E. D.

PROP. VI.

In any plane triangle ABC, it will be, as the base
 plus the difference of the two sides, is to the base
 minus the same difference, so is the tangent of half
 the greater angle at the base, to the tangent of half
 the lesser.



In the lesser side CA,
 produced, take $CD =$
 CB , so that AD may be
 the difference of the two
 sides; and let BD be
 drawn: then it is mani-
 fest that the angle CBD
 will be equal to D: but

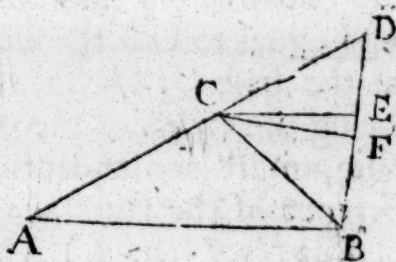
(*by Theor. 5. p. 8.*) $AB + AD : AB - AD ::$
 $\text{tang. } \frac{D + DBA}{2} = \frac{CAB}{2}$ (*by 9. 1.*) : tang.
 $\frac{D - DBA}{2} = \frac{CBD - DBA}{2} = \frac{CBA}{2}.$ Q. E. D.

PROP.

PROP. VII.

As the base of any plane triangle ABC, is to the sum of the two sides, so is the sine of half the vertical angle, to the co-sine of half the difference of the angles at the base,

In AC, produced, take $CD = CB$; join B, D, and draw CE parallel to AB, and CF perpendicular to BD.



Since $CD = CB$, therefore is the angle $D = DBC$ (by 12. 1.) and, consequently, half the vertical angle $ACB = \frac{D + CBD}{2}$ (by 9. 1.) $= D$.

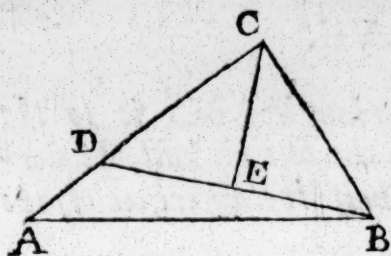
Moreover, seeing DCB is $=$ the sum of the angles A and CBA , at the base (by 9. 1.) it is evident that BCF (or DCF) is equal to half that sum; and, therefore, as ECF is the excess of the greater $ABC = BCE$ (by 7. 1.) above the half sum (BCF), it must, manifestly, be equal to half the difference of the same angles A and CBA .

But (by Theor. 3.) $AB : AD (AC + BC) :: \text{fine } D (\frac{1}{2}ACB) : \text{fine } ABD = \text{fine } CED$ (by Cor. 1. to 7. 1.) $= \text{fine } FEC = \text{co-sine } ECF$. Q. E. D.

PROP. VIII.

As the base of any plane triangle ABC, is to the difference of the two sides, so is the co-sine of half the vertical angle, to the sine of half the difference of the angles at the base.

In

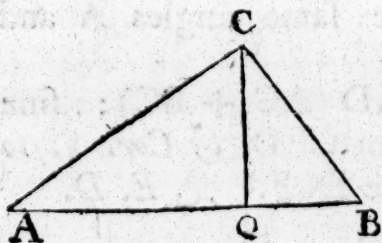


In the greater side CA let there be taken $CD = CB$, and let BD be drawn, and likewise CE , perpendicular to BD . It is manifest, because $CD = CB$, that CDB and CBD are equal to one another, and that each of them is also equal to half the sum of the angles CBA and A at the base (*by Cor. 2. to 10. 1.*); therefore ABD , being the excess of the greater CBA above the half sum, must consequently be equal to half the difference of the same angles.

But (*by Theor. 3.*) $AB : AD (AC - BC) :: \text{fine } D (\text{co-sine } DCE, \text{ or } \frac{1}{2}C) : \text{fine } ABD.$ *Q. E. D.*

PROP. IX.

As the difference of the two sides AC, BC, of a plane triangle, is to the difference of the segments of the base AQ, BQ (made by letting fall a perpendicular from the vertex), so is the sine of half the vertical angle, to the co-sine of half the difference of the angles at the base.



For, $AC - BC : AQ - BQ :: AB : AC + BC$
(by 9. 2. and 10. 4.) $:: \text{fine } \frac{ACB}{2} : \text{co-sine } \frac{B - A}{2}$
(by Prop. 7.) *Q. E. D.*

PROP. X.

As the sum of the two sides of a plane triangle, is to the difference of the segment of the base (see the preceding figure), so is the co-sine of half the vertical angle, to the sine of half the difference of the angles at the base.

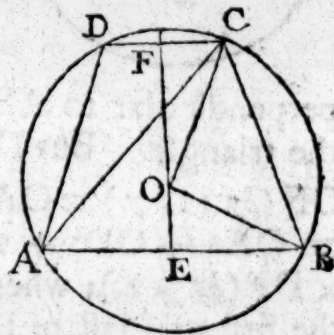
For,

For, $AC + BC : AQ - BQ :: AB : AC - BC$
 (by 9. 2. and 10. 4.) $:: \text{co-sine of } \frac{ACB}{2} : \text{sine of}$
 $\frac{B - A}{2}$ (by Prop. 8.) Q. E. D.

PROP. XI.

As the tangent of the vertical angle C of a plane triangle ABC, is to radius, so is half the base AB to a fourth-proportional; and as half the base is to the excess of the perpendicular above the said fourth-proportional, so is the sine of the vertical angle, to the co-sine of the difference of the angles at the base.

Let ABCD be a circle described about the triangle, and from O, the center thereof, let OB and OC be drawn; moreover, draw CD parallel to BA, meeting the periphery in D, and EOF, perpendicular to AB, meeting DC in E. Then it is evident, that EF will be equal to the perpendicular height of the triangle, EO equal to the vertical angle ACB, and FOC (= DAC) equal to the difference of the angles (ABC and BAC) at the base.

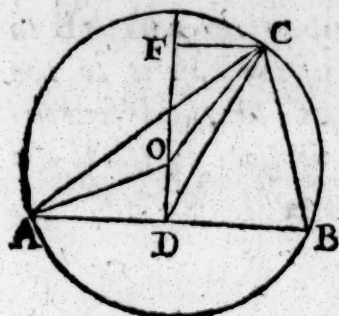


But (by Theor. 2.) as tang. EOB (ACB) : radius $:: EB (\frac{1}{2}AB) : EO$; moreover, as $EB : OF (EF - EO) :: \text{sine EOB (ACB)} : \text{sine OCF}$, or co-sine of FOC. Q. E. D.

PROP.

PROP. XII.

As the tangent of the vertical angle, of a plane triangle ABC, is to radius, so is the base AB to a fourth-proportional; and, as the said fourth-proportional, is to the sum of the semi-base and the line CD bisecting the base, so is the difference of these two, to the perpendicular height of the triangle.



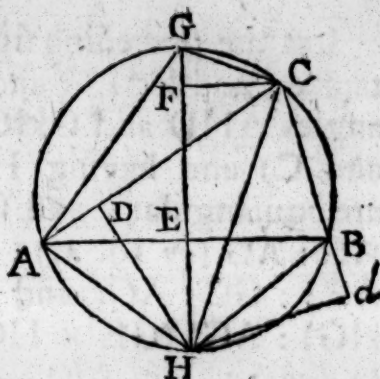
Let a circle be described about the triangle, and from O, the center thereof, let OA, OC and OD be drawn; also let CF, parallel to AB, be drawn, meeting DO, produced (if need be) in F. It is evident that DF will be perpendicular to AB, and equal to the height of the triangle. But $DC^2 = OC^2 + OD^2 + 2OD \times OF$ (by 11. 2.) $= OA^2 + OD^2 + 2OD \times DF - OD^2 = OA^2 - OD^2 + 2OD \times DF = AD^2 + 2OD \times DF$ (by 7. 2.); whence, by taking away AD^2 from the first and last of these equal quantities, we have $DC^2 - AD^2 = 2OD \times DF$; or $DC + AD \times DC - AD = 2OD \times DF$ (by 7. 2.) and therefore $2OD : DC + AD :: DC - AD : DF$; but (by Theor. 2.) as the tang. AOD = ACB (by 10. 3.): to radius ($AD : OD$) :: AB : 2OD. Q. E. D.

PROP. XIII.

As twice the rectangle under the base and perpendicular of a plane triangle ABC, is to the rectangle under the sum, and difference, of the base and sum of the two sides; so is radius, to the co-tangent of half the vertical angle.

Let

Let HG , perpendicular to AB , be the diameter of a circle described about the triangle; and let HD and Hd be perpendicular to the two sides of the triangle; also let CF be parallel to AB , and let HA , HB and HC be drawn.



Since the diameter HG is perpendicular to AB , therefore is $AE = BE$ (by 2. 3.) $AH = BH$, and the angle $ACH = BCH$ (by 12. 3.); whence, also, $CD = Cd$, and $HD = Hd$ (by 15. 1.) Therefore, the right-angled triangles HAD and HBd , having $AH = HB$ and $HD = Hd$, have, likewise, $AD = Bd$ (by 15. 1.) From whence it is manifest, that CD will be equal to half the sum, and AD equal to half the difference, of the two sides of the triangle. Moreover, because of the similar triangles AEH and HCD , it will be, $AE^2 : CD^2 :: HA^2 = HE \times HG$ (by Cor. to 19. 4.) $: HC^2$ ($HF \times HG$) $:: HE : HF$ (by 7. 4.) Whence, by division, &c. $AE^2 : CD^2 - AE^2 :: HE : EF :: HE \times AE : EF \times AE$; therefore, by inversion and alternation, $EF \times AE : CD^2 - AE^2$ ($CD + AE \times CD - AE$) $:: HE \times AE : AE^2$. $HE : AE :: \text{radius} : \text{co-tang. EAH}$ (ACH , by Theor. 2.): whence the truth of the proposition is manifest.

PROP. XIV.

As twice the rectangle of the base and perpendicular of a plane triangle ABC , is to the rectangle under the sum, and difference, of the base and the difference of the two sides; so is radius, to the tangent of half the vertical angle.

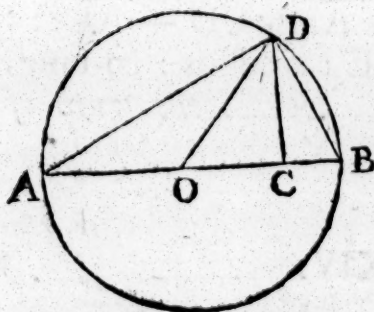
F

Let

Let the preceding figure and construction be retained, and let AG and CG be drawn. The triangles AHD and GHC (being right-angled at D and C) and having $\angle HAD = \angle HGC$ (by 11. 3.) are equiangular; and so $AD : GC :: AH : HG :: AE : AG$ (by 19. 4.); whence, alternately, $AD : AE :: GC : AG$, and $AD^2 : AE^2 :: GC^2 (GF \times HG) : AG^2 (GE \times HG) :: GF : GE$; therefore, by division, &c. $AE^2 - AD^2 : AE^2 :: EF : GE :: EF \times AE : GE \times AE$; whence, again, by alternation, &c. $EF \times AE : AE^2 - AD^2 (AE + AD \times AE - AD) :: GE \times AE : AE^2 :: GE : AE :: \text{radius} : \text{tang. AGE}$ (by Theor. 2.); from which the truth of the proposition is manifest.

PROP. XV.

If the relation of three right-lines a , b and x , be such, that $ax - x^2 = b^2$; then it will be, as $\frac{1}{2}a : b :: \text{radius to the sine of an angle; and, as radius, to the tangent (or co-tangent) of half this angle, so is } b : x$.



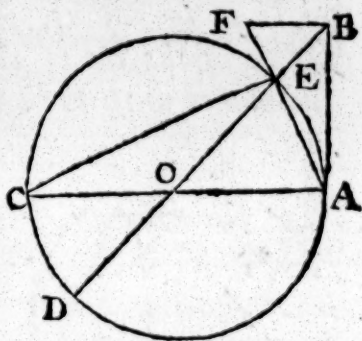
Make AB equal to a , upon which let a semi-circle ADB be described; also let CD, equal to b , be perpendicular to AB, and meet the periphery in D (for it cannot exceed the radius of the circle when the proposition is possible): moreover, let AD, BD, and the radius OD, be drawn. Because $AC \times CB = CD^2 = b^2$ (by Cor. to 19. 4.) it is plain that $AC \times a - AC$, or $BC \times a - BC$ is also $= b^2$; and, therefore, $x \times a - x$ being $= b^2$, it is manifest that x may be equal, either, to AC, or to BC. Now
(by

(by Theor. 1.) $OD (\frac{1}{2}a) : CD (b) :: \text{radius} : \text{fine } DOC$; whose half is equal to A, or BDC (by 10. 3.) But, as $\text{radius} : \text{tang. } BDC :: DC (b) : BC$; or, as $\text{radius} : \text{co-tang. } BDC (\text{tang. } CDA) :: DC (b) : AC$. Q. E. D.

PROP. XVI.

If the relation of three lines, a , b and x , be such, that $x^2 \pm ax = b^2$; then it will be, as $\frac{1}{2}a : b :: \text{radius to the tangent of an angle; and as radius is to the tangent, or co-tangent, of half this angle (according as the sign of } ax \text{ is positive or negative)} :: b : x$.

Make $AB = b$, and AC , perpendicular to AB , equal to a ; about the latter of which, as a diameter, let a circle be described; and, thro' O , the center thereof, let BD be drawn, meeting the periphery in E and D ; also let A, E and C, E be joined, and draw BF parallel to AC , meeting AE , produced, in F . Then, since (by 22. 3.) $BE \times BD (= BE \times BE + a = BD \times BD - a) = AB^2 (b^2)$ and $x \times x \pm a = b^2$, by supposition, it is manifest, that BE will be $= x$, when $x \times x + a = b^2$; and $BD = x$, when $x \times x - a = b^2$.



Furthermore, because the angle $F = OAE$ (by 7. 1.) $= OEA$ (by 12. 1.) $= BEF$ (by 3. 1.) it is evident that $BF = BE$ (by 18. 1.) and that the angles BAF and C (being the complements of the equal angles F and OAE) are likewise equal.

Now (by Theor. 2.) $AO (\frac{1}{2}a) : AB (b) :: \text{radius} : \text{tang. } AOB$; whose half is equal to C , or BAF (by 14. 3.) But, as $\text{radius} : \text{tang. } BAF :: AB (b)$

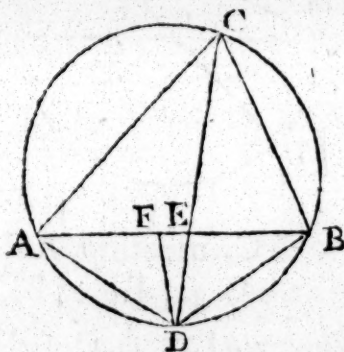
F 2

: BF

: BF (BE), the value of x in the first case, where $x^2 + ax = b^2$. Again, radius : co-tang. BAF (tang. F) :: BF (BE) : AB (by Theor. 2.); and BE : AB :: AB : BD (by Cor. to 22. 3. and 10. 4.); whence, by equality, radius : co-tang. BAF :: AB (b) : BD; which is the value of x in the second case; where $x^2 - ax = b^2$. Q. E. D.

PROP. XVII.

In any plane triangle ABC, it will be, as the line CE bisecting the vertical angle, is to the base AB, so is the secant of half the vertical angle ACB, to the tangent of an angle; and, as the tangent of half this angle is to radius, so is the sine of half the vertical angle, to the sine of either angle, which the bisecting line makes with the base.



Let ACBD be a circle described about the triangle, and let CE be produced to meet the periphery thereof in D; moreover, let AD and BD be drawn, and likewise DF, perpendicular to the base AB; which will, also, bisect it, because (BCD being = ACD) the subtenses BD and AD are equal (by 10. 3.) Moreover, since the angle DBE = ACD (by 12. 3.) = DCB, the triangles DEB and DBC (having D common) are equiangular, and therefore $DE \times DC = DB^2$ (by 24. 3.) or, which is the same, $DE^2 + DE \times CE = DB^2$. Therefore (by Prop. 16.) $\frac{1}{2}CE : DB :: \text{radius} : \text{tangent of an angle (which we will call Q)} ; \text{ and, as radius : tang. } \frac{1}{2}Q :: DB : DE.$

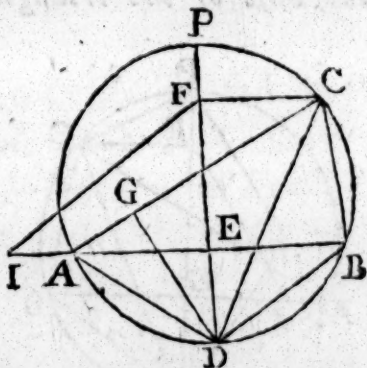
But $DB : BF (\frac{1}{2}AB :: \text{secant FBD (BCE)} : \text{radius} ; \text{ therefore, by compounding this, with the first$

first proportion, we have, $\frac{1}{2}CE \times DB : \frac{1}{2}AB \times DB :: \text{radius} \times \text{secant BCE} : \text{radius} \times \text{tang. Q}$ (by 10. 4.) and consequently $CE : AB :: \text{secant BCE} : \text{tang. Q}$ (by 7. 4.) Again, $DE : DB :: \text{fine DBE (BCE)} : \text{fine of DEB (or of CEB)}$; whence, by equality, $\text{tang. } \frac{1}{2}Q : \text{radius} :: \text{fine BCE} : \text{fine CEB}$. Q. E. D.

PROP. XVIII.

In any plane triangle ABC, it will be, as the perpendicular is to the sum of the two sides, so is the tangent of half the angle at the vertex, to the tangent of an angle; and, as radius is to the tangent of half this angle, so is the sum of the two sides, to the base of the triangle.

Let DP, perpendicular to AB, be the diameter of a circle described about the triangle; let CF be perpendicular to DP, and DG to AC, and let DA, DC and DB be drawn, and also FI, parallel to BD, meeting BA, produced, in I.

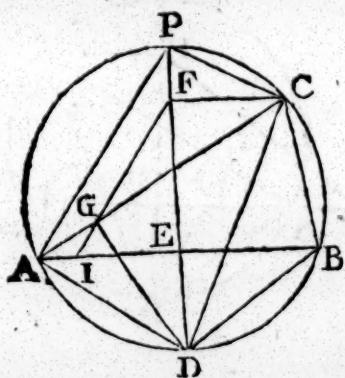


It is manifest, from Prop. 13. that CG is equal to half the sum of the sides AC and BC: it also appears, (from 7. 1. and Cor. to 12. 3.) that the triangles DCG, ADE, BDE, and IFE, are all equiangular. Therefore it will be, $CG^2 : BE^2 :: DC^2 (DF \times DP) : BD^2 (DE \times DP) :: DF (DE + EF) : DE :: BE + EI : EB :: \overline{BE + EI} \times BE : EB^2$; and, consequently, $CG^2 (= \overline{BE + EI} \times BE) = BE^2 + EI \times BE$. But, by Prop. 16. it will be, $\frac{1}{2}EI : CG :: \text{radius} : \text{tangent of an angle (which we will call Q)}$; and as $\text{radius} : \text{tang. } \frac{1}{2}Q :: CG : BE$

($\therefore 2CG$, or $AC + BC$, to AB). But (*by Theor. 2.*) $EF : EI :: \text{tang. } I. (ACD) : \text{radius}$; therefore, by compounding this proportion with the last but one, we shall have, $\frac{1}{2}EI \times EF : EI \times CG :: \text{tang. } ACD \times \text{radius} : \text{tang. } Q \times \text{radius}$ (*by 11. 4.*) and consequently $EF : 2CG (AC + BC) :: \text{tang. } ACD : \text{tang. } Q$. Whence the truth of the proposition is manifest.

PROP. XIX.

In any plane triangle ABC , it will be, as the perpendicular is to the difference of the two sides, so is the co-tangent of half the vertical angle, to the tangent of an angle; and, as radius is to the co-tangent of half this angle, so is the difference of the sides to the base of the triangle.



Let $DP, DG, CF, \&c.$ be as in the preceding proposition; also let PA and PC be drawn, and FI , parallel to PA , meeting AB in I .

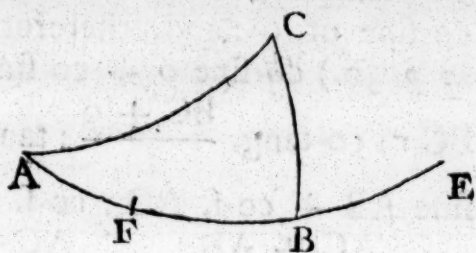
The right-angled triangles ADG and DPC , having $DAG = DPC$ (*by Cor. to 12. 3.*) are similar; and therefore, $AG^2 : PC^2 :: AD^2 : DP^2 :: AE^2 : AP^2$ (*by Cor. to 11. 4.*); whence, alternately, $AG^2 : AE^2 :: PC^2 (PF \times PD) : AP^2 (PE \times PD) :: PF : PE :: AI : AE :: AI \times AE : AE^2$ (*by 7. 4.*); and consequently, $AG^2 = AI \times AE = AE^2 - EI \times AE$. Therefore, *by Prop. 16*, $\frac{1}{2}EI : AG :: \text{radius} : \text{tangent of an angle } (Q)$; and as $\text{radius} : \text{co-tang. } \frac{1}{2}Q :: AG : AE$. But (*by Theor. 2.*) $EF : EI :: \text{co-tang. } EFI (ACD) : \text{radius}$; which proportion being compounded with the last but

but one, &c. we shall have, $EF : 2 AG (AC - BC,$
see Prop. 13.) :: co-tang. EFI : tang. Q; and as
 radius : co-tang. $\frac{1}{2}Q :: AG : AE :: 2AG (AC -$
 $BC) : 2AE (AB).$ Q. E. D.

PROP. XX.

The hypotenuse AC, and the sum, or difference, of the legs AB, BC, of a right-angled spherical triangle ABC, being given, to determine the triangle.

Let AE be the sum, and AF the difference of the two legs. Because, radius : co-s. AB :: co-s. BC : co-s. AC (by Theor. 2.)



therefore, $\text{co-s. } AB \times \text{co-s. } BC = \text{rad.} \times \text{co-s. } AC$; but the former of these is $= \frac{1}{2} \text{rad.} \times \text{co-s. } AE + \text{co-s. } AF$ (by Corol. 3. to Prop. 2.); therefore $2 \times \text{co-s. } AC = \text{co-s. } AE + \text{co-s. } AF$. Whence it appears, that, if from twice the co-sine of the hypotenuse, the co-sine of the given sum, or difference, of the legs, be subtracted, the remainder will be the co-sine of an arch which added to the said sum, or difference, gives the double of the greater leg required.

COROLLARY.

Hence, if the two legs be supposed equal to each other (or the given difference $= 0$), then will the co-sine of the double of each, be equal to twice the co-sine of the hypotenuse minus the radius.

PROP. XXI.

One leg BC and the sum, or difference, of the hypotenuse and the other leg AB being given, to determine the hypotenuse (see the last figure.)

Since $\text{rad.} : \text{co-sine BC} :: \text{co-sine AB} : \text{co-sine AC}$ (by Theor. 2.), it will be (by comp. and div.) $\text{radius} + \text{co-sine BC} : \text{rad.} - \text{co-sine BC} :: \text{co-sine AB} + \text{co-sine AC} : \text{co-sine AB} - \text{co-sine AC}$. But the radius may be considered as the sine of an arch of 90° , or the co-sine of 0 : and, therefore, since (by the lemma in p. 30.) $\text{co-sine } 0 + \text{co-sine BC} : \text{co-sine } 0 - \text{co-sine BC} :: \text{co-tang. } \frac{\text{BC} + 0}{2} : \text{tang. } \frac{\text{BC} - 0}{2}$; and, $\text{co-sine AB} + \text{co-sine AC} : \text{co-sine AB} - \text{co-sine AC} :: \text{co-tang. } \frac{\text{AC} + \text{AB}}{2} : \text{tang. } \frac{\text{AC} - \text{AB}}{2}$; it follows, by equality, that $\text{co-tang. } \frac{\text{BC}}{2} : \text{tang. } \frac{\text{BC}}{2} :: \text{co-tang. } \frac{\text{AC} + \text{AB}}{2} : \text{tang. } \frac{\text{AC} - \text{AB}}{2}$; that is, *As the co-tang. of half the given leg, is to its tangent; so is the co-tang. of half the sum of the hypotenuse and the other leg, to the tangent of half their difference.*

PROP. XXII.

The angle at the base and the sum, or difference, of the hypotenuse and base, of a right-angled spherical triangle being given, to determine the triangle.

First,

First, it will be, $\text{rad.} :$
 $\text{co-f. } A :: T. AC : T. AB$
 (by Theor. 1.) and therefore
 $\text{rad.} + \text{co-f. } A : \text{rad.} -$
 $\text{co-f. } A :: T. AC + T.$
 $AB : T. AC - T. AB :$

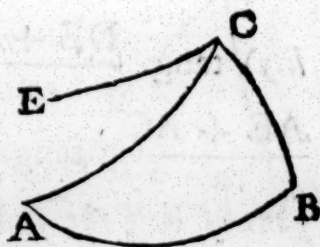


whence, by arguing as in the last Prop. it will appear, that, $\text{co-tang. } \frac{1}{2}A : \text{tang. } \frac{1}{2}A :: \text{rad.} +$
 $\text{co-f. } A : \text{rad.} - \text{co-f. } A (:: T. AC + T. AB : T.$
 $AC - T. AB) :: S. AC + AB : S. AC - AB$ (by
 Prop. 4.). Hence it appears, that, *As the co-tan-*
gent of half the given angle, is to its tangent; so is
the sine of the sum of the hypotenuse and adjacent
leg, to the sine of their difference.

PROP. XXIII.

The hypotenuse AC and the sum, or difference, of the two adjacent angles being given, to find the angles.

Let EC be perpendicular
 to BC; and then it will be,
 $\text{rad.} : \text{co-f. } AC :: T. A :$
 $T. ACE$ (by Theor. 5.)
 From whence, by reasoning
 as above, we shall, also,
 have, $\text{co-tang. } \frac{1}{2}AC : \text{tang.}$



$\frac{1}{2}AC :: S. A + ACE : S. A - ACE$; whereof the
 two last terms, by substituting $90^\circ - ACB$ for
 ACE , will become $S. 90^\circ + A - ACB$ ($\text{co-f. } ACB$
 $- A$) and $S. A + ACB - 90^\circ$ respectively. Whence
 it appears, that, *As the co-tangent of half the hy-*
pothenuse, is to its tangent; so is the co-sine of the dif-
ference of the angles at the hypotenuse, to the sine of
the excess of their sum above a right-angle.

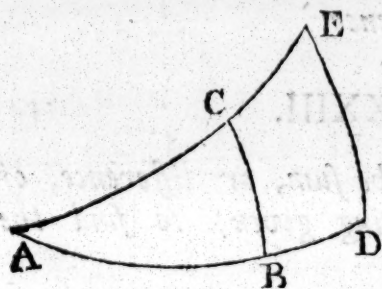
COROL-

COROLLARY.

Hence, if the angles be supposed equal, then it will be, as radius : tang. $\frac{1}{2}AC$:: tang. $\frac{1}{2}AC$; sin. $2A - 90^\circ$.

PROP. XXIV.

In two right-angled spherical triangles ABC, ADE, having one angle A common, let there be given the two perpendiculars BC, DE and the sum, or difference, of the hypotenuses AC, AE, to determine the triangles.



It is evident (from Theor. 1.) that S. DE : S. BC :: S. AE. S. AC ; therefore S. DE + S. BC : S. DE - S. BC :: S. AE + S. AC : S. AE - S. AC : whence (by the lemma in p. 30. and equality)

tan. $\frac{DE + BC}{2}$: tan. $\frac{DE - BC}{2}$:: tan. $\frac{AE + AC}{2}$: tan. $\frac{AE - AC}{2}$: that is, As the tangent of half the sum of the two perpendiculars, is to the tangent of half their difference ; so is the tangent of half the sum of the two hypotenuses, to the tangent of half their difference.

PROP. XXV.

In two right-angled spherical triangles ABC, ADE, having the same angle A at the base, let there be given the two perpendiculars BC, DE and the sum or difference of the bases AB, AD, to determine the bases (see the preceding figure.)

Since

Since $T. DE : T. BC :: S. AD : S. AB$ (by Theor. 4. and equality); therefore is $T. DE + T. BC : T. DE - T. BC :: S. AD + S. AB : S. AD - S. AB$; whence; (by Prop. 4. and the lemma in p. 30.) it will be, $S. DE + BC : S. DE - BC :: T. AD + AB : T. AD - AB$; that is,

2

2

As the sine of the sum of the two perpendiculars, is to the sine of their difference; so is the tangent of half the sum of the two bases, to the tangent of half their difference.

PROP. XXVI.

The product of the square of radius and the co-sine of the base of any spherical triangle ABC, is equal to the product of the sines of the two sides and the co-sine of the vertical angle, together with the product of radius and the co-sines of the same sides.

For let AD be perpendicular to BC; then, since co-f. BD =

$$\frac{S. CB \times S. CD + co-f. CB \times co-f. CD}{rad.}$$

(by Cor. 1. to Prop. 2.) it is evident, that co-f. BD : co-f. CD ::

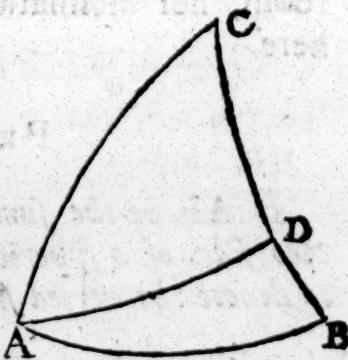
$$\frac{S. CB \times S. CD + co-f. CB \times co-f. CD}{rad.}$$

$$: co-f. CD :: \frac{S. CB \times S. CD}{co-f. CD} + co-f. CB : rad. \text{ (by}$$

$$\text{mult. each term by } \frac{rad.}{co-f. CD} \text{). But } \frac{S. CD}{co-f. CD} = \frac{T. CD}{rad.},$$

$$\text{therefore our proportion will be } \frac{S. CB \times T. CD}{rad.} +$$

co-f.



co-f. CB : rad. (:: co-f. BD : co-f. CD) :: co-f. AB : co-f. AC (*by Corol. to Theor. 2.*) whence, by multiplying means and extremes, we have co-f.

$$AB \times \text{radius} = \frac{S. CB \times \text{co-f. AC} \times T. CD}{\text{rad.}} +$$

co-f. AC $\times$ co-f. BC. But (*by Theor. 1.*) radius :

$$\text{co-f. C} :: T. AC : T. CD = \frac{\text{co-f. C} \times T. AC}{\text{rad.}} =$$

$$\frac{\text{co-f. C} \times S. AC}{\text{co-f. AC}} \text{ (*by Corol. 1. Prop. 1.*) which last}$$

being substituted for its equal, we shall have,

$$\text{co-f. AB} \times \text{rad.} = \frac{S. CA \times S. CB \times \text{co-f. C}}{\text{rad.}} +$$

co-f. AC $\times$ co-f. BC ; from whence, if each term be multiplied by radius, the truth of the proposition will appear manifest.

There is another way of demonstrating this proposition, from the orthographic projection of the sphere ; but that is a subject which neither room, nor inclination, will permit me to treat of here.

PROP. XXVII.

If AE be the sum, and AF the difference, of the two sides of a spherical triangle ABC, and V be put to denote the versed sine of the vertical angle, and R

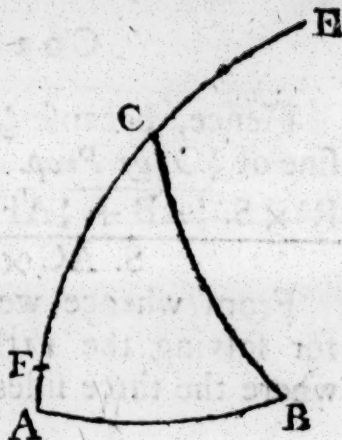
the radius ; then will $V = \frac{R^2 \times \text{co-f. AF} - \text{co-f. AB}}{S. AC \times S. BC}$

$$= \frac{2R \times \text{co-f. AF} - \text{co-f. AB}}{\text{co-f. AF} - \text{co-f. AE}}$$

$$= \frac{2R \times S. \frac{1}{2}AB + \frac{1}{2}AF \times S. \frac{1}{2}AB - \frac{1}{2}AF}{S. AC \times S. BC}$$

It appears from the last Prop. that $\text{co-f. } AB \times R$ is $= \text{co-f. } AC \times \text{co-f. } BC + \frac{S. AC \times S. BC \times \text{co-f. } C}{R}$;

in which, for $\text{co-f. } C$ let its equal $R - V$ be substituted, and then we shall have $\text{co-fine } AB \times R = \text{co-fine } AC \times \text{co-fine } BC + \text{fine } AC \times \text{fine } BC - \frac{S. AC \times S. BC \times V}{R}$: but the



sum of the two former of the three last terms is $= \text{co-f. } AF \times R$ (by Cor. 1. to Prop. 2.); therefore it will be $\text{co-f. } AB \times R = \text{co-f. } AF \times R - \frac{S. AC \times S. BC \times V}{R}$, and consequently $V =$

$$\frac{R^2 \times \text{co-f. } AF - \text{co-f. } AB}{S. AC \times S. BC}.$$

Which is the first case. Again, because $S. AC \times S. BC$ is $= \frac{1}{2}R \times \text{co-f. } AF - \text{co-f. } AE$ (by Corol. 3. to Prop. 2.) we shall, also, have $V = \frac{2R \times \text{co-f. } AF - \text{co-f. } AB}{\text{co-f. } AF - \text{co-f. } AE}$.

Which is the second case. Moreover, since $\frac{1}{2}R \times \text{co-f. } AF - \text{co-f. } AB$ is $= S. \frac{AB + AF}{2} \times S.$

$$\frac{AB - AF}{2} \text{ (by the same) it follows that } V \text{ is likewise}$$

$$= \frac{2R \times S. \frac{1}{2}AB + \frac{1}{2}AF \times S. \frac{1}{2}AB - \frac{1}{2}AF.}{S. AC \times S. BC}$$

Q. E. D.

COROLLARY 1.

Hence, because $\frac{1}{2}R \times V$ is = the square of the sine of $\frac{1}{2}C$ (by Prop. 1.) it follows that $\text{sq. S. } \frac{1}{2}C = \frac{R^2 \times \text{S. } \frac{1}{2}AB + \frac{1}{2}AF \times \text{S. } \frac{1}{2}AB - \frac{1}{2}AF}{\text{S. AC} \times \text{S. BC}}$.

From whence we have the following theorem, for solving the 11th case of oblique triangles, where the three sides are given, to find an angle.

As the rectangle of the sines of the two sides, including the proposed angle, is to the rectangle under the sines, of half the base plus half the difference of the sides, and half the base minus half the difference of the sides; so is the square of radius, to the square of the sine of half the required angle.

COROLLARY 2.

Moreover, because V is $= \frac{R^2 \times \text{co-f. AF} - \text{co-f. AB}}{\text{S. AC} \times \text{S. BC}}$, we shall have $R^2 : \text{S. AC} \times \text{S. BC} :: V : \text{co-f. AF} - \text{co-f. AB}$; which gives the following theorem, for finding a side, when the opposite angle, and the other two sides, are given.

As the square of radius, is to the rectangle of the sines of the two sides including the given angle; so is the versed sine of that angle, to the difference of the co-sines (or versed sines) of the difference of those sides, and the side required.

* COROL.

COROLLARY 3.

Lastly, because $V = \frac{2R \times \text{co-f. AF} - \text{co-f. AB}}{\text{co-f. AF} - \text{co-f. AE}}$,
 we shall, by transforming the equation, and putting W for $(2R - V)$ the versed sine of BCE (the supplement of the vertical angle) have $\text{co-f. AE} = \frac{2R \times \text{co-f. AB} - W \times \text{co-f. AF}}{V}$, and the
 $\text{co-sine AF} = \frac{2R \times \text{co-f. AB} - V \times \text{co-f. AE}}{W}$.

From whence the sides themselves may be determined, when their sum, or difference, is given, with the base and vertical angle.

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